Asymmetries in Monetary Policy

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March 17, 2021

Abstract

Nonlinearities embedded in the standard New-Keynesian model show that a welfare-maximizing policymaker should behave in line with a contractionary bias, fearing more expansions in output and inflation rather than contractions. On the contrary, the aggregate-supply equation implies that any upward pressure coming from real marginal costs does not necessarily push up inflation. Once these two forces are combined in the optimal policy, an overall expansionary bias emerges. The nonlinearities of the AS equation combined with changes in volatility can be responsible for a flattening in the estimated linear Phillips curve.

1 Introduction

The most popular framework used for monetary policy analysis is built on objectives for output and inflation that are usually symmetric with respect to targets and on a linear model economy, as in Galì (2008) and Woodford (2003). Output and inflation fluctuate symmetrically around their steady-state levels and prescriptions for what monetary policy should do are identical irrespective of whether the economy is experiencing expansions or contractions.

Recent literature has shown that macroeconomic variables such as output, inflation and unemployment display some skewness and that business cycles can be asymmetric both in the size and duration of expansions and contractions, e.g. Dupraz et al. (2019) and Salgado et al. (2019). Moreover, central banks around the world repeatedly miss their inflation target since inflation constantly proves to be on a lower level. This also suggests that there could be some bias in their preferences, nonlinearity in the model economy or asymmetries in the shocks hitting the economy.[1]

Motivated by these facts, this paper studies asymmetries in monetary policy by uncovering non-linear effects behind the standard New-Keynesian model. To this end, it exploits a cubic approximation of welfare and a quadratic approximation of the model economy, as opposed to the standard quadratic-linear framework of Benigno and Woodford (2003), Woodford (2003) and the related literature.

The results are the following: A policymaker maximizing consumers' welfare should fear more expansions in output and inflation rather than contractions. Therefore, preferences show a contractionary bias. On the contrary, by accounting for non-linear effects, the aggregate-supply equation implies that any upward pressure coming from real marginal costs does not necessarily result in an upward pressure on inflation. Therefore, there could be a natural tendency for the economy to display a deflationary bias. Once these two forces are combined in the optimal policy, an overall expansionary bias emerges implying a relatively higher inflation following both positive or negative mark-up shocks with respect to what would have been implied by the standard linear-quadratic analysis.

Finally, we use our framework to provide a possible explanation for the flattening of the U.S. Phillips curve. In our analysis, this can be due to an omitted-variable problem when the data generating process features some nonlinearities. An econometrician estimating a linear Phillips curve would indeed omit all second order terms which could be responsible for the flattening of the curve in periods of high volatility, something that has been observed in the data after the great financial crisis. Indeed, our non-linear (second-order approximated) New-Keynesian aggregate-supply equation shows that current and past deviations of inflation with respect to the target can weaken the inflation-output trade-off. Bygones are not bygones. If an economy is hit by large shocks pushing inflation below target, a larger output gap is required to produce the same movements in inflation had inflation, instead, always been on target.

This paper is related to recent literature that has investigated asymmetries in the business cycle. Dupraz et al. (2019) argue for a plucking model of the business cycle based on downward wage rigidities to explain a skewed empirical distribution of unemployment. The

[1] Bec et al. (2003) have found asymmetries in the monetary policy reaction function even in the past.

New-Keynesian literature has also underlined possible asymmetries in policies due to the zero-lower bound, as in Eggertsson and Woodford (2003), and in combination with downward-wage rigidities, as in Coibion et al. (2012). Our analysis is complementary to all these works since we study nonlinearities already built into the standard framework rather than adding other asymmetries that potentially could work to amplify our results.

Castillo and Montoro (2008) is a closely related work since they use a second-order approximation of a standard New Keynesian model to study the asymmetric response of output and inflation, but only to an interest-rate shock. To account for the evidence showing that monetary policy is more effective in upturns, they also assume nonhomotheticity in preferences. Castillo et al. (2005), instead, uses a second-order approximation of the standard New-Keynesian model to explain the path of inflation through oil shocks.

None of the above works has analyzed optimal monetary policy in a non-linear environment like we do. The only exception is the work of Gross and Hansen (2020), who provided a general theory of quadratic-cubic approximations, applying it to a different version of the New Keynesian model and emphasizing the importance of asymmetries in wage rigidities. Their work and ours have been conducted in an independent way.[2]

Moreover, we are also contributing to the literature on optimal targeting rules spurred by the work of Giannoni and Woodford (2017) since we have extended their analysis to a cubic-quadratic approximation, showing how second-order terms can affect the standard linear targeting rule discussed in the literature, as in Svensson (1999).

Finally, our paper is related to the recent literature that has investigated the possible causes of the flattening in the Phillips curve by suggesting an alternative and complementary explanation to those given in the literature (see among others Blanchard (2016), Coibion and Gorodnichenko (2012) and Hazell et al. (2020)). Coibion and Gorodnichenko (2012) argue that the missing disinflation following the great financial crisis could be explained by a rise in inflation expectations between 2009 and 2011. Blanchard (2016) and Hazell et al. (2020) find a modest decline in the Phillips curve in the last two decades and attribute the stability of inflation to a firm anchoring of inflation expectations. Our analysis suggests that estimates of the Phillips curve could be downward biased when volatility increases, since in this case the relation between real marginal costs and inflation weakens when nonlinearities are important.

Our paper is structured as follows. Section 2 presents the model. Section 3 studies the asymmetries in monetary policy resulting from preferences, from the aggregate-supply equation and from the optimal targeting rule, respectively. Section 4 studies the optimal asymmetric policy following mark-up shocks. Section 5 compares the path of inflation and output gap in the U.S. economy with the counterfactual, in which optimal policy is conducted using either the quadratic-cubic approximation or the standard linear-quadratic approximation. Section 2 investigates the possible flattening of the Phillips curve because of omitted second-order terms in the estimation. Section 7 concludes the paper.

[2] As soon as we became aware of their work, at the end of October 2020, we corresponded with them by sending our draft paper, which was similar to the current version, but with preliminary ideas and results only for the current Sections 5 and 6.

2 Model

We present our analysis via the benchmark New Keynesian model. Here we outline the building blocks of the model by referring to the literature for a more exhaustive treatment.[3] A representative agent maximizes expected intertemporal utility

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\ln C_t - \frac{N_t^{1+\phi}}{1+\phi} \right] \tag{1}$$

in which β, with $0 < \beta < 1$, is the intertemporal discount factor, C_t is a consumption basket of goods and N_t is hours worked; ϕ, with $\phi > 0$, is the inverse of the Frisch elasticity of labor supply. The consumption bundle C_t is a Dixit-Stiglitz aggregator of a continuum of goods, with measure one, produced in the economy. The elasticity of substitution among these goods is σ. Financial markets are assumed to be complete and the representative agent chooses consumption, labor and takes portfolio decisions to maximize its utility under an appropriately-defined budget constraint and borrowing limit.

The first-order conditions of the household's optimization problem imply an Euler equation, which links current and future consumption to the real rate. Using equilibrium in the goods markets, i.e. consumption C_t is equal to aggregate output Y_t, the Euler equation can be written as

$$Y_t^{-\rho} = \beta(1+i_t) E_t \left\{ \frac{P_t}{P_{t+1}} Y_{t+1}^{-\rho} \right\} \tag{2}$$

where P_t is the consumption-based price index and i_t is the risk-free nominal interest rate. Labor supply entails equalization of the marginal rate of substitution between consumption and labor to the real wage.

Turning to the supply side, there is a continuum of firms, each producing one of the variety of goods consumed in the economy. A generic firm j produces goods using the technology $y_t(j) = A_t N_t(j)$, where A_t is a productivity shock common to all firms. Firms sell goods in a market characterized by monopolistic competition, facing the demand $y_t(j) = (p_t(j)/P_t)^{-\sigma} Y_t$, where $p_t(j)$ is the price of the variety of good j. The price-setting mechanism follows the Calvo model in which a fraction $(1-\alpha)$ of firms is randomly selected to change their prices independently of the last time they reset them. In each period all prices are adjusted to the (gross) inflation target Π. Given the preference specification assumed, the supply side of the model implies an aggregate supply equation of the form

$$\left(\frac{1 - \alpha \left(\frac{\Pi_t}{\Pi} \right)^{\sigma-1}}{1-\alpha} \right)^{\frac{1}{\sigma-1}} = \frac{E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left(\frac{P_T}{P_t \Pi^{T-t}} \right)^{\sigma-1}}{E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \mu_T \frac{N_T^{1+\phi}}{\Delta_T} \left(\frac{P_T}{P_t \Pi^{T-t}} \right)^{\sigma}}, \tag{3}$$

where Π_t is the gross inflation rate, defined as $\Pi_t \equiv P_t/P_{t-1}$; μ_t is a mark-up shock whose variations depend on changes in distortionary taxes levied on firms' labor costs; Δ_t is an index of price dispersion defined by

$$\Delta_t \equiv \int_0^1 \left(\frac{p_t(j)}{P_t} \right)^{-\sigma} dj,$$

[3] See Galì (2008) and Woodford (2003).

which follows the law of motion

$$\Delta_t \equiv \alpha\Delta_{t-1}\left(\frac{\Pi_t}{\Pi}\right)^{\sigma} + (1-\alpha)\left(\frac{1-\alpha\left(\frac{\Pi_t}{\Pi}\right)^{\sigma-1}}{1-\alpha}\right)^{\frac{\sigma}{\sigma-1}}. \tag{4}$$

We can also express the AS equation (3) in terms of output by noting the relationship between aggregate employment and output

$$N_t = \int_0^1 N_t(j)dj = \frac{1}{A_t}\int_0^1 y_t(j)dj = \frac{Y_t}{A_t}\Delta_t. \tag{5}$$

The model then consists of an aggregate demand equation (2) and an aggregate supply equation (3), which together with the law of motion of price dispersion and a monetary-policy rule determine the path of inflation, output and interest rates given the two stochastic disturbances: a productivity and a mark-up shock.

3 Asymmetries in monetary policy

This Section is divided into three parts: the first studies asymmetries built into the welfare function; the second in the aggregate-supply equation; the third in the optimal targeting rule.

3.1 Asymmetries in welfare

We are interested in evaluating policies according to the welfare of the representative agent following the Ramsey approach to optimal policy. To account for nonlinearities in the response of the macroeconomic variables, an approximation of optimal policy to an order higher than the first is needed: a second-order approximation is sufficient.

The solution of the Ramsey optimal policy problem is, in general, time-inconsistent when the constraints of the problem contain expectations on future variables. However, it can become time-consistent if additional commitments are considered at time zero, as in the timeless-perspective approach of Woodford (2003). In this case, Benigno and Woodford (2013) has shown the equivalence between the solution obtained by maximizing the appropriate quadratic approximation of welfare under a linear approximation of the constraints and the one obtained by just linearizing the first-order conditions of the optimal policy problem. However, such a solution does not display any role for asymmetric responses of the variables of interest to shock or even a distinct role for volatility shocks. Indeed, in a quadratic approximation of welfare, deviations from targets in the objective function are equally penalized independently of the sign.

The results of Benigno and Woodford (2013) can still be helpful for our analysis since they can be extended to higher-order approximations. In our case, a second-order approximation of the optimal policy problem can be equivalently obtained as the solution of a cubic-quadratic approximation method in which an appropriate third-order approximation of welfare is maximized under a second-order approximation of the constraints.[4] In this Section we use this insight to get an idea of the shape of the objective and constraints, and postpone the numerical analysis to Section 4.

We make a simplifying assumption, common to the literature, in approximating the model around the efficient steady state, an assumption that we maintain throughout the paper. First, note that we can write (1) as

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \ln(N_t A_t) - \frac{N_t^{1+\phi}}{1+\phi} - \ln \Delta_t \right\} \tag{6}$$

in which we have used equilibrium in the goods market, $Y_t = C_{t,}$ and (5) to substitute Y_t for N_t.

Taking a third-order log-linear approximation of the above utility around the efficient steady state, and disregarding terms independent of policy and the higher-order ones, we get the following intertemporal loss function

$$L_{t_o} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\frac{1}{2} n_t^2 + \frac{1}{6}(1+\phi) n_t^3 + \frac{\ln \Delta_t}{1+\phi} \right] \tag{7}$$

in which n_t is at the same time the log of the employment level and the employment gap, i.e. the deviations of the log of employment with respect to the efficient level. To see that $\tilde{n}_t = \ln \tilde{N}_t = 0$ is the efficient employment level, note that in the efficient allocation the marginal rate of substitution between consumption and labor is equal to productivity:

$$\frac{\tilde{N}_t^{\phi}}{\tilde{C}_t^{-1}} = A_t,$$

which, given equilibrium in goods market, $Y_t = C_t$, and the linear production technology, $Y_t = A_t N_t$, implies that $\tilde{N}_t = 1$. The important novelty shown by a third-order approximation is the additional cubic term in the employment gap, as shown by the loss function (7). It implies that positive deviations of employment from the efficient level are more costly than negative ones. Once nonlinearities are considered, a policymaker maximizing households' welfare is subject to a contractionary bias and is averse to expansions, being more penalized by increases in employment rather than falls. The last term in the approximation of the utility captures the costs of price dispersion, which are always non-negative since $\Delta_t \geq 1$.

The loss function (7) can also be expressed in terms of the output gap, y_t, noting that $y_t = n_t - \Delta_t$.[5] Since Δ_t is at least a second-order term in the norm of the shock, the loss function can be written – disregarding terms of order higher than the third – as

$$L_{t_o} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\frac{1}{2}(y_t + \Delta_t)^2 + \frac{1}{6}(1+\phi) y_t^3 + \frac{\ln \Delta_t}{1+\phi} \right]. \tag{8}$$

Comparing the above loss function with the standard quadratic one in the literature, it already displays a slightly different form by just looking at quadratic terms, since Δ_t appears

[4] Gross and Hansen (2020) show this equivalence formally.

[5] Note that with log utility the efficient level of output is given by $Y_t = A_t$.

in the first, quadratic, term on the right-hand side. It can indeed be neglected in a second-order approximation but not in a third-order approximation. The price dispersion term is non-negative and is function of the current and past squared deviations of inflation from the target, as will be shown shortly. This implies that deviations of inflation from the target tilt the output-gap target to negative values. Moreover, the second term on the right-hand side of (8), the cubic term, further reinforces the aversion to expansions built into preferences.

We now turn to characterizing the third-order approximation of the price dispersion term defined in (4). In the Appendix we show that it is given by

$$\hat{\Delta}_t = \alpha\hat{\Delta}_{t-1} + \frac{1}{2}\frac{\sigma\alpha}{1-\alpha}(\pi_t - \pi)^2 + \alpha\sigma\hat{\Delta}_{t-1}(\pi_t - \pi) + \frac{1}{6}\frac{\sigma\alpha}{1-\alpha}\gamma(\pi_t - \pi)^3, \qquad (9)$$

where γ is defined as

$$\gamma \equiv \frac{(\sigma - 1)}{(1-\alpha)} + \sigma - \frac{\alpha}{1-\alpha},$$

which is in general positive, at least for $\sigma \geq 2$. We have used the following definitions $\pi_t \equiv \ln P_t/P_{t-1}$ and $\pi \equiv \ln \Pi$. Note that price dispersion is zero up to first-order terms and depends on the squared deviations of inflation with respect to the target only when looking at second-order terms. These results are known in the literature. The third-order terms are instead captured by the last two addenda on the right-hand side of equation (9). Note, indeed, that $\hat{\Delta}_t$ is at least of a second order in the norm of the shocks and therefore $\hat{\Delta}_{t-1}(\pi_t - \pi)$ is a third-order term. In general, upward movements of inflation with respect to the target contribute positively to price dispersion in contrast to downward movements. When inserting (9) into the welfare function, this asymmetry results in a general aversion to overshooting the inflation target. We then obtain

$$L_{t_o} = E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left\{\frac{1}{2}n_t^2 + \frac{1}{6}(1+\phi)n_t^3 + \frac{1}{2}\frac{\sigma}{\kappa}(\pi_t - \pi)^2 + \frac{\sigma}{\kappa}(1-\alpha)\hat{\Delta}_{t-1}(\pi_t - \pi) + \frac{1}{6}\frac{\sigma}{\kappa}\gamma(\pi_t - \pi)^3\right\}. \qquad (10)$$

Accounting for asymmetries in the loss function leads to important novelties should a benevolent policymaker care about upward or downward deviations of employment or inflation from the target. What is implicit in the above micro-founded loss function is therefore an aversion to overshooting both the efficient level of employment and the inflation target. Thus, according to this welfare-based loss function, a policymaker should behave following a contractionary bias.

3.2 Asymmetries in the AS equation

Evaluating optimal policy requires understanding the trade-off between employment and inflation implicit in the AS equation (3). To evaluate a third-order approximation of welfare, a second-order approximation of the constraints is sufficient. Moreover, this approximation can show asymmetries in the trade-off that are not present in the standard linear approximation.

In the Appendix, following Benigno and Woodford (2003), we show that a second-order approximation of (3) delivers the following set of equations

$$V_t + (1-\alpha)\hat{\Delta}_{t-1} = \kappa n_t + u_t + \frac{\kappa}{2}(n_t + u_t)^2 + \beta E_t[V_{t+1} + (1-\alpha)\hat{\Delta}_t] \qquad (11)$$

where κ is given by

$$\kappa \equiv \frac{1-\alpha}{\alpha}(1-\alpha\beta)(1+\phi)$$

and u_t is a reparametrization of the mark-up shock with $u_t \equiv \kappa\hat{\mu}_t/(1+\phi)$.

Note that V_t is defined by the following equation

$$V_t \equiv (\pi_t - \pi) + \frac{1}{2}\left[1 + \sigma\frac{\alpha}{1-\alpha}\right](\pi_t - \pi)^2 + (\sigma - 1)(\pi_t - \pi)X_t \tag{12}$$

with X_t following

$$X_t = (\pi_t - \pi) + \alpha\beta E_t X_{t+1}. \tag{13}$$

To get an idea of the AS relationship first note that firms' real marginal costs are given by

$$mc_t = \frac{W_t}{A_t P_t} = \mu_t N_t^{1+\phi}$$

and that the deviation of the gross inflation rate from the steady state, in a second-order approximation, is given by

$$\frac{\Pi_t - \Pi}{\Pi} = (\pi_t - \pi) + \frac{1}{2}(\pi_t - \pi)^2.$$

Using these observations, we can write (11) as

$$\frac{\Pi_t - \Pi}{\Pi} = kE_t\left\{\sum_{T=t}^{\infty}\beta^{T-t}\frac{mc_T - mc}{mc}\right\} - (1-\alpha)\hat{\Delta}_{t-1} - \frac{\sigma}{2}\frac{\alpha}{1-\alpha}(\pi_t - \pi)^2 - (\sigma - 1)(\pi_t - \pi)X_t,$$

for some parameter k and where mc is the steady-state real marginal cost. The first term on the right-hand side of the above equation is in line with the literature, which mainly relies on first-order approximations, saying that deviations of inflation from the steady state are explained by deviations of the real marginal cost from its steady state. Factors that push up the real marginal cost, as a mark-up shock or an increase in the employment gap, lead to a rise of inflation above the target. However, a second-order approximation shows additional terms affecting the inflation rate. The first term, which is the second on the right-hand side of the above equation, shows that past deviations of inflation from the target in either directions produce a downward pressure on current inflation. Indeed, note that up to a second-order approximation, price dispersion follows the law of motion

$$\hat{\Delta}_t = \alpha\hat{\Delta}_{t-1} + \frac{1}{2}\frac{\sigma\alpha}{1-\alpha}(\pi_t - \pi)^2. \tag{14}$$

The third term on the right-hand side also adds downward pressure on current inflation if the latter deviates from the target. Finally, the last term can also reduce current inflation if there is a positive correlation between the current deviations of inflation from target and its expected present-discounted value captured by the term X_t. Overall, these additional terms, which can only be uncovered by a second-order approximation of the AS equation, imply that any upward pressure coming from real marginal costs does not necessarily result in an upward pressure on inflation. There could be a disinflationary bias coming from past, present and future deviations of inflation from the target. Interestingly, the AS equation shows that missing the inflation target in the past leads to downward pressures on current inflation, making it even harder to achieve the target, unless there is more pressure coming from real marginal costs. Bygones are not bygones.

3.3 Asymmetries in the optimal targeting rule

Two interesting results have emerged from the previous Sections. First, asymmetries in the loss function show a contractionary bias both with respect to inflation and the employment gap. Second, asymmetries in the AS equation show that the relationship between real marginal costs and inflation can become weaker and that deviations of inflation from target can put a downward pressure on inflation. Another way to obtain these results is by having the contractionary bias in the loss function be lessened by the disinflationary pressure built in the AS equation. What policymakers should then do can only be understood by solving the optimal policy problem.

As already discussed, a second-order approximation of the optimal policy can be obtained by minimizing the loss function (10) under constraints (11), (12), (13) and (14).

A first result can be directly seen without taking first-order conditions. Following productivity shocks, it is optimal to set inflation to target at all times and stabilize employment and output at their efficient level. This confirms the findings in the literature. Trade-offs arise only when there are mark-up shocks.

An intuitive way to study this trade-off is to derive the optimal targeting rule, as in Giannoni and Woodford (2017). In the Appendix, we show that it takes the following form

$$\sigma(\pi_t - \pi) + (y_t - y_{t-1}) = \mathcal{T}_t \tag{15}$$

which looks very similar to the one they obtain in a linear-quadratic framework, except for the additional term on the right-hand side, $\mathcal{T}_t$, which is indeed zero in a first-order approximation. With a zero $\mathcal{T}_t$, an overshoot of the inflation target is optimal provided the output gap falls from the previous period. A non-zero value of $\mathcal{T}_t$ can change this result in interesting directions. A positive $\mathcal{T}_t$ can either mitigate the output-gap contraction or allow for more expansion in inflation. It then leads to an expansionary bias. On the contrary, a negative $\mathcal{T}_t$ requires a larger fall in output for a given overshoot of inflation with respect to its target. It acts as a contractionary bias.

In the Appendix we show that $\mathcal{T}_t$ can be decomposed into five components

$$\mathcal{T}_t = \tau_1(\pi_t - \pi)^2 + \tau_2(\pi_t - \pi)E_t X_{t+1} + \tau_3\hat{\Delta}_{t-1} + \tau_4(y_t^2 - y_{t-1}^2) + \tau_5(y_t u_t - y_{t-1}u_{t-1}), \tag{16}$$

in which τ_1, τ_2, τ_3, τ_4, τ_5 are all positive parameters defined in the Appendix. In what follows, we label each component as $\mathcal{T}_{j,t}$ for $j = 1, ..., 5$ with $\mathcal{T}_{1,t} \equiv \tau_1(\pi_t - \pi)^2$, $\mathcal{T}_{2,t} \equiv \tau_2(\pi_t - \pi)E_t X_{t+1}$, $\mathcal{T}_{3,t} \equiv \tau_3\hat{\Delta}_{t-1}$, $\mathcal{T}_{4,t} \equiv \tau_4(y_t^2 - y_{t-1}^2)$ and $\mathcal{T}_{5,t} \equiv \tau_5(y_t u_t - y_{t-1}u_{t-1})$. The first three components capture the contemporaneous, future and past second-order effects of inflation on the targeting rule. The fourth and fifth components capture the second-order effects of marginal costs and mark-up shocks. We now describe each component in turn. The first term is always positive insofar as there are deviations of inflation from the target, therefore implying an expansionary bias. As we have seen, the AS equation shows a weaker trade-off between inflation and output gap when inflation deviates from the target, and this can be exploited when setting the optimal response of inflation to shocks. The expansionary bias built into the AS equation dominates the contractionary bias coming from preferences. A positive $\mathcal{T}_t$ in (15) can accommodate a more expansionary response of inflation or output. The second term reflects the correlation between the current deviation of inflation from the target and the expected present discounted value of those deviations, captured by X_{t+1}. If these comovements are positive, an expansionary bias will emerge. This effect is implied, as well, by the form of the AS equation. The third term captures instead the effects of past inflation. Even in this case, past deviations of inflation from the target imply a weaker trade-off between inflation and the output gap, which can be exploited to run a more expansionary policy. The fourth term captures the effects of deviations of the square of output gap with respect to the previous level. Any increase leads to a more expansionary bias. Finally, the last term captures the effects of shocks in combination with the output gap and their previous levels. Suppose that at time t the economy is hit by a cost-push shock. The output gap falls on impact and the combined effect therefore leads to a contractionary bias which can be accommodated through lower inflation or less reduction in the output gap.

4 Optimal asymmetric response to shocks

In this Section, we combine previous results to determine the optimal response to mark-up shocks. Figure 1 shows the optimal response to a positive mark-up shock by comparing the first-order approximation with a second-order approximation. In the numerical analysis, the discount factor β is calibrated to 0.99. The other parameters are estimated using the procedure we detail in the Appendix. Namely, the inverse of the Frisch elasticity is $\phi = 0.2$, the fraction of firms that do not reset their price in the Calvo model is $\alpha = 0.904$, while the elasticity of substitution among intermediate goods is set at $\sigma = 4.8$.

Figure 1 shows that a positive cost-push shock increases inflation and reduces the output gap.[6] Inflation remains positive because of the persistence of the shock, but eventually falls below the target value and converges to it from below. Output shows a hump-shaped response, converging very lately to the steady-state value. A second-order approximation differs from the first order along some dimensions. Inflation, output gap and interest rates are higher once accounting for nonlinearities.

The second and third charts in the second row display the deviations from the targeting rule, the variable $\mathcal{T}_t$ of equation (15) and its decomposition into the five components of (16). The variable $\mathcal{T}_t$ is always positive: an expansionary bias arises which is reflected by a relatively higher inflation, as we have discussed. Two are the components of the decomposition that matter more for the deviations of the targeting rule from the zero benchmark of the first-order approximation. The second term, $\mathcal{T}_{2,t}$, capturing the future movements of inflation explains the short-run positive value of $\mathcal{T}_t$. The first term $\mathcal{T}_{1,t}$, driven by the current deviations of inflation from the target, explains instead the persistence in the expansionary bias.

Figure 2 repeats the same experiment but for a negative mark-up shock. In comparison with Figure 1, results are no longer symmetric except for the case of log-linear approximations in which the response is exactly specular.

A second-order approximation is instead characterized by an expansionary bias, which mostly results in a higher path of inflation that substantially overshoots the target. This is reflected by a positive value of $\mathcal{T}_t$ in (15), which is of the same magnitude as that of

[6]The magnitude of the shock is 0.0022, which is 20 times the estimated standard deviation. This large shock is needed to appreciate the differences in the impulse responses, as it is done in the literature. See, among others, Basu and Bundick (2017).

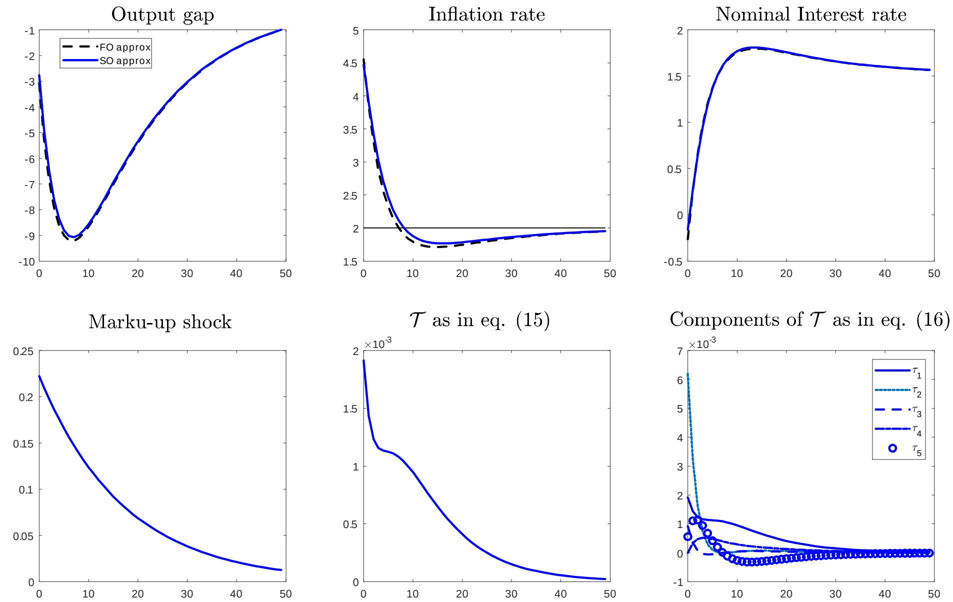

Figure 1: Impulse responses to a positive innovation to the mark-up process of output gap, inflation, nominal interest rate, mark-up, $\mathcal{T}$ given by equation (15) and its components given by equation (16). Second-order approximation (blue solid line) versus first-order approximation (black dashed line). Inflation and interest rates are in % and at annual rates. Output gap is in %.

Figure 2. Note that $\mathcal{T}_t$ can be appropriately evaluated using a first-order approximation and, in this approximation, responses are specular and of similar magnitude. Therefore, all the components of $\mathcal{T}_t$ in (16) have the same sign and magnitude. We can then summarize the results. Once accounting for second-order terms and following either positive or negative mark-up shocks, optimal policy requires more accommodation in inflation without sacrificing the output gap.

5 Asymmetries in inflation and output

As shown in the previous Section, a non-linear analysis uncovers the possible asymmetries in the response of the economy to shocks, which, therefore, can generate asymmetries in the distribution of inflation and output. In this Section, we further explore the implications of these asymmetries by running a thought experiment on U.S. data to compare the implications of the linear model versus the non-linear (second-order approximated) model.

Let us first consider the linear model and the related New-Keynesian AS equation. Assume that this equation is data consistent for appropriately calibrated/estimated parameters. Under this assumption and by using the data path of inflation and output gap, it is possible to use the AS equation to back up a path for the mark-up shock. Details of the procedure are left to the Appendix.[7] Given the filtered mark-up series, we can then ask which paths

[7]The series of the quarterly real GDP (labeled GDPC1_NBD19470101 in the FRED database) and the quarterly real gross potential output (labeled GDPPOT_NBD19490101), together with the series of the

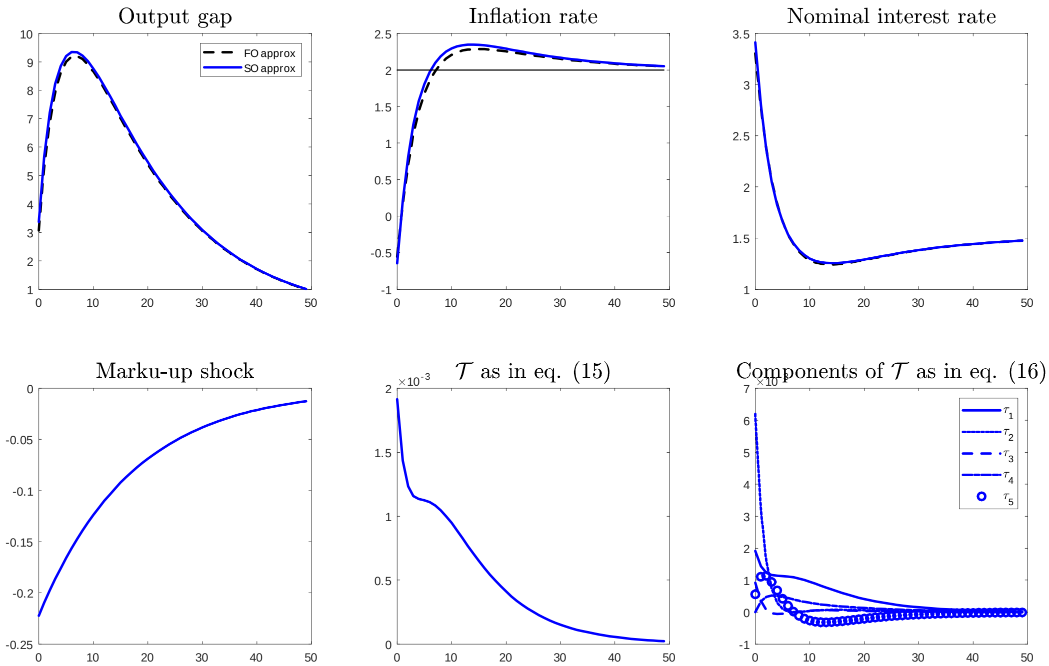

Figure 2: Impulse responses to a negative innovation to the mark-up process of output gap, inflation, nominal interest rate, mark-up, $\mathcal{T}$ given by equation (15) and its components given by equation (16). Second-order approximation (blue solid line) versus first-order approximation (black dashed line). Inflation and interest rates are in % and at annual rates. Output gap is in %.

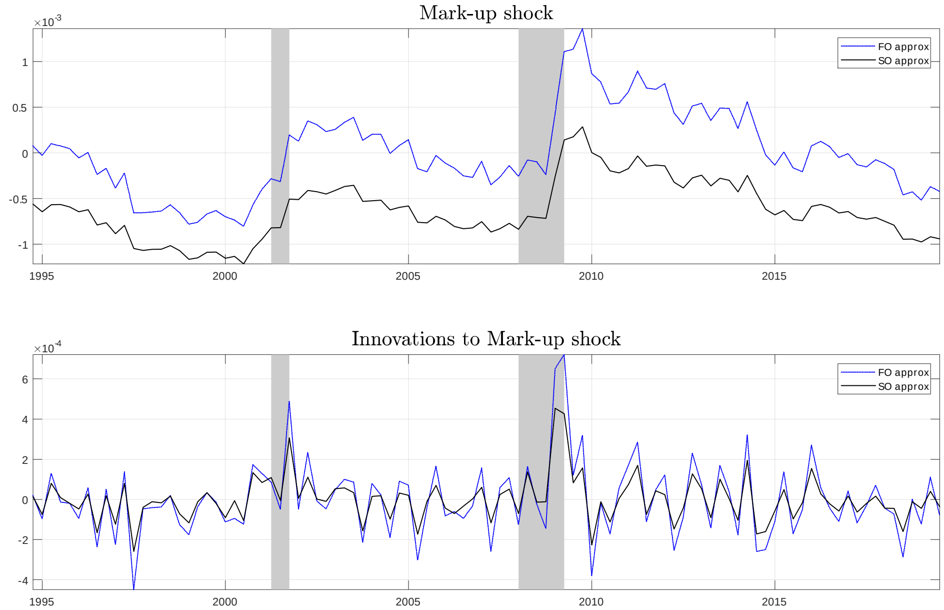

Figure 3: Plot of the filtered mark-up series and its innovation using the first-order approximation model (FO approx) and the second-order approximation model (SO approx).

of inflation and output would have occurred were policies conducted optimally using the quadratic-linear model. We repeat the same experiment using the cubic-quadratic model. First, using the second-order approximation of the AS equation, we filter the path of mark-up shock consistent with the data path of inflation and output gap. Then, we compute the path of inflation and output gap under the optimal cubic-quadratic model. Note that the two experiments are aligned to be equivalent in replicating the data on inflation and output with the respective filtered mark-up series and the respective AS equation. Optimal policy will instead be different.

Figure 3 shows the filtered paths of the mark-up shocks, modelled as AR(1) processes, and their innovations implied by using the linear model (the line labelled "FO approx") and the non-linear model (the line labelled "SO approx"), respectively. The estimated persistence of the process is 0.920 in the case of the linear AS equation and 0.943 in the other case, while standard deviations of the innovations are 0.0185/100 and 0.0111/100, respectively.

Figure 4 compares the paths of inflation, output gap and its growth computed through the two optimal policy problems with the paths seen in the data. In describing the Figure, we first underline the differences between the data and the two optimal-policy experiments and then we dissect the differences between the two optimal policies. First, note that inflation is more volatile in the data than under optimal policies, in which cases it remains centered around the 2% inflation target. Focusing on the two recessions marked by the grey areas,

quarterly core CPI inflation index (labeled CPILFESL) are downloaded from the FRED database for the period 1995Q1-2019Q3. The series of the output gap is computed by taking the difference between the logarithm of Real Gross Domestic Product and that of Real Potential Gross Domestic Product. The series of inflation is instead obtained by taking the quarterly log-difference of the core CPI.

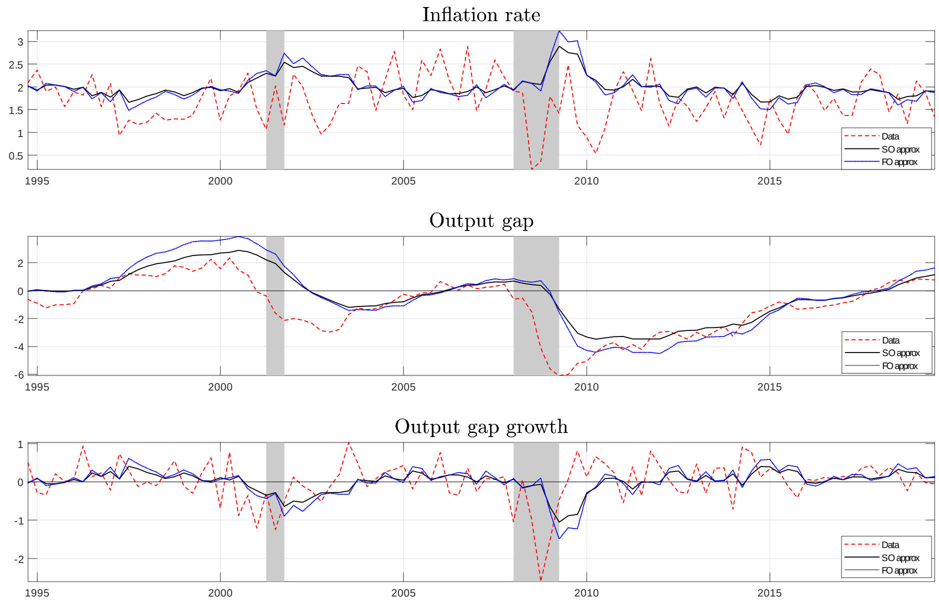

Figure 4: Plot of inflation, output gap and output gap growth. Comparison among data (red-dashed line), optimal policy using first-order approximation (FO) (blue line) and second-order approximation (SO) (black line). Inflation is in % and at annual rate, output gap is in %.

it is interesting to note that optimal policy would have implied an inflation rate well above the two-percent target in contrast with the below-target and subdued inflation rate seen in the data. This more extended accommodation would have implied a prolonged expansion in output without, however, preventing its fall, albeit delayed until 2010 and of a smaller magnitude than what is seen in the data. After 2010, the recovery in the output gap under optimal policies reflects that of the data.

Turning to the comparison between the two optimal policies, we observe some differences in the optimal inflation rate mainly during the recession periods in which the optimal inflation rate is higher under the first-order approximation than in the second-order approximated model. This difference comes with important benefits for output growth under the second-order approximated model. In particular, after the 2007-2008 financial crisis, the output gap falls less if optimal policy follows the quadratic-cubic model.

The first panel of Figure 5 displays the difference between the two optimal targeting rules, captured by the term $\mathcal{T}_t$ in (15). The decomposition of this difference is plotted over time in the bottom panel of the Figure according to the split given by the five components identified in (16). The first striking result is that $\mathcal{T}_t$ is always positive in the sample and spikes in the aftermath of the recessions. Remember that a positive value of $\mathcal{T}_t$ allows inflation to overshoot the target without requiring a proportional fall in the output gap. But, what are the drivers of the spikes? The first component, $\mathcal{T}_{1,t} = \tau_1(\pi_t - \pi)^2$, is the dominating one. As we have seen, the AS equation shows a weaker trade-off between inflation and output gap when inflation deviates from the target. Optimal policy requires inflation to overshoot the target, which at the same time creates a weaker trade-off between inflation and output gap,

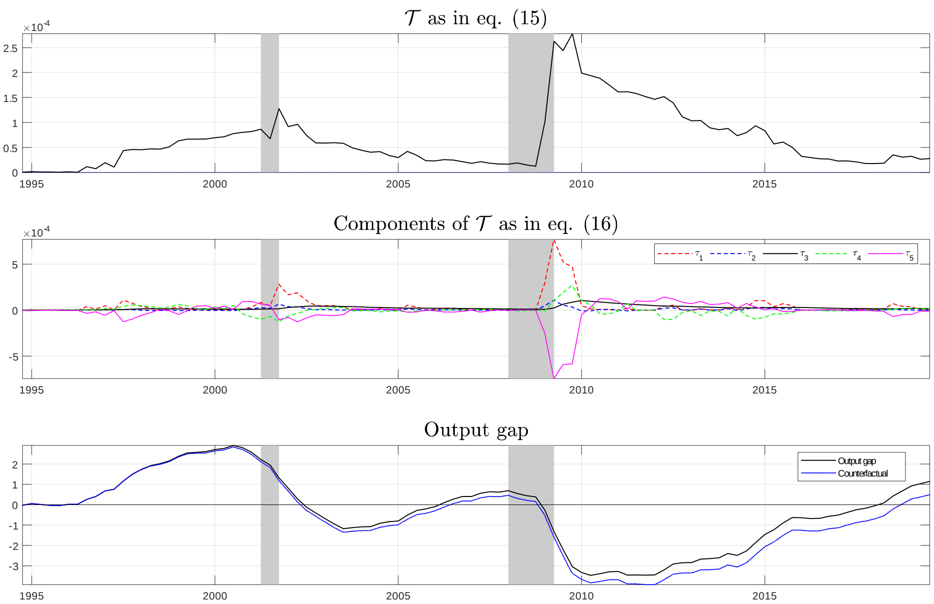

Figure 5: First and second panel: plots of $\mathcal{T}$ of equation (15) and its components $\mathcal{T}_j$ for $j = 1, 2, ..., 5$ given by (16). Third panel: plot of the output gap under optimal policy using the second-order approximation (black line) and counterfactual output gap (blue dashed line) implied by using the targeting rule $\sigma(\pi_t - \pi) + (y_t - y_{t-1}) = 0$ in which the inflation path is the same as in the optimal policy under a second-order approximation. Inflation is in % and at annual rate, output gap growth is in %.

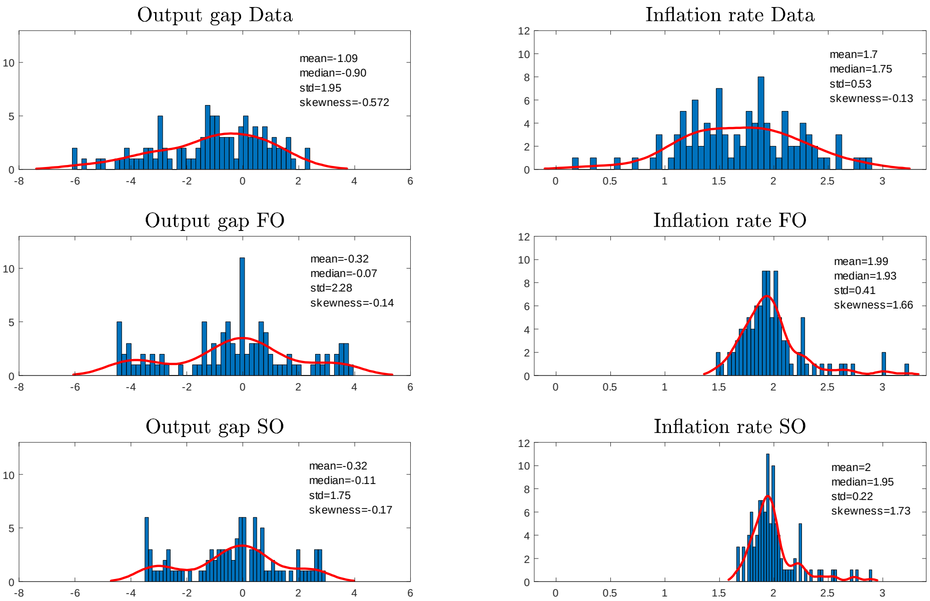

Figure 6: Histograms of output gap (first column) and inflation (second column). Comparison among data (first row), optimal policy under first-order approximation (FO) (second row), optimal policy under second-order approximation (SO) (third row). Mean, median, standard deviation and skewness are reported in each panel for the respective figure. Inflation is in % and at annual rates, output gap is in %.

allowing output gap to fall by less. In both spike episodes this channel is partly offset by the fifth component, while in the aftermath of the second recession of the sample it is reinforced by the third component. The last panel of the Figure compares the path of the output gap under the optimal policy by using the second-order approximated model with that obtained using the targeting rule $\sigma(\pi_t - \pi) + (y_t - y_{t-1}) = 0$ in which the inflation rate coincides with that under the same optimal policy. The figure allows us to capture in a better way the expansionary bias implied by a positive $\mathcal{T}_t$ in the optimal targeting rule (15).

Finally, Figure 6 reports the histograms of inflation and output gap, comparing their data values with those implied by the two optimal policies. Whereas inflation, in the data, is not skewed and output is left skewed, under optimal policies inflation is right skewed and output is not skewed. Inflation in the data has a mean of around 1.7% at annual rates while it is centered at around 2% under optimal policies, and it is slightly higher under the optimal policy computed by using the second-order approximation. In this case it is also marginally more skewed and less volatile. Output gap is instead more dispersed under the two optimal policies, and it is symmetric. It has a higher mean and median, and it is also less volatile under the optimal policy when using the second-order approximated model.

This last Figure conveys an interesting message: had the policymakers been following an optimal policy, we would have observed smaller deviations of inflation from the target with a more skewed distribution of inflation displaying a larger number of observations above the 2% target. The less volatile inflation would have come without any sacrifice in terms of average output gap.

6 AS Estimation

We now use our model to address a recent macroeconomic puzzle which has been extensively discussed in literature and policy circles: the flattening of the Phillips curve. Inflation in the U.S. economy has been running below target notwithstanding the fall in unemployment and the growing economy, at least until the COVID-19 pandemic. Our model can provide a possible explanation along the following lines. If the data generating process is the non-linear model, an econometrician who instead estimates a linear AS equation could face an omitted-variable problem, which can be responsible for an overly too low estimation of the slope of the equation. Note, indeed, that in Section 3.2 we emphasized that accounting for second-order terms implies a downward pressure on inflation given real marginal costs. The educated guess therefore is that higher volatility could play a role in estimating a lower slope of the Phillips curve.

To evaluate this conjecture we simulate the second-order approximation of the model considered in the previous section. We use the same calibration of the parameters as in the previous Section, the same estimated parameters of the bivariate VAR used to characterize the dynamic of the output gap and the same "estimated" process of the mark-up shock filtered by using the model to match the data. Then, we use the simulated data on inflation and output gap to estimate the following linear Phillips curve relating inflation and output gap using 5.000 samples of 250 quarters each

$$\pi_t = a + by_t + \varepsilon_t. \tag{17}$$

	Benchmark	Case 2σ	Case 5σ	Case 10σ
a	-0.0005	-0.002	-0.014	-0.056
	(-0.001, 0.000)	(-0.004, -0.001)	(-0.016, -0.009)	(-0.072, -0.041)
y_t	0.268	0.262	0.252	0.241
	(0.23, 0.30)	(0.22, 0.30)	(0.20, 0.30)	(0.16, 0.32)
R^2	0.494	0.461	0.289	0.1409

Table 1: Estimates of equation (17). Cases 2σ, 5σ, 10σ consider respectively 2, 5, 10 times the standard deviation of the benchmark case. Confidence intervals are reported below each estimate. The coefficients and the standard deviations are the median values of the 5.000 estimations.

We repeat the same procedure by increasing the standard deviation of all the shocks by the same magnitude, respectively two, five and ten times larger than in the benchmark case. Table 1 reports the results. The median value of the coefficient of the output gap, b, is positive and it decreases as the standard deviation of the shocks rises. Note that values of b close to 0.2 are consistent with the estimates found in the data, as shown in Blanchard (2016). The conjecture that an increase in volatility could have lowered the estimates of the linear AS equation, in an otherwise non-linear world, seems validated by this analysis.

We repeat the same experiment by estimating the New-Keynesian Phillips curve

$$\pi_t = a' + b' y_t + \beta E_t \pi_{t+1} + \varepsilon_t, \tag{18}$$

in which the one-period ahead inflation expectations are computed by using a bivariate VAR on the simulated inflation and output-gap data. Table 2 reports the results.

	Benchmark	Case 2σ	Case 5σ	Case 10σ
a'	-0.000 (-0.0004, 0.002)	-0.000 (-0.001 0.000)	-0.002 (-0.005 0.000)	-0.009 (-0.016 -0.000)
y_t	0.053 (0.030 0.076)	0.051 (0.026 0.075)	0.044 (0.012 0.076)	0.040 (-0.008 0.089)
$E_t \pi_{t+1}$	0.897 (0.83 0.96)	0.905 (0.840 0.970)	0.933 (0.860 1.004)	0.943 (0.861 1.019)
R^2	0.887	0.872	0.803	0.747

Table 2: Estimates of equation (18). Cases 2σ, 5σ, 10σ consider respectively 2, 5, 10 times the standard deviation of the benchmark case. Confidence intervals are reported below each estimate. The coefficients and the standard deviations are the median values of the 5.000 estimations.

Also in this case, the coefficient of the output gap decreases as the standard deviation of the shocks rises. The estimate of the coefficient in front of the one-period ahead inflation expectations is not far from the calibrated one, 0.99, and it increases with the standard deviations of the shocks. For a robustness check, we repeat the same estimation using one-period ahead inflation expectations derived from the estimation of AR(1) and AR(2) processes on the simulated inflation data. The results are quantitatively and qualitatively similar to those of Table 2. Thus, we do not report them in the main text.

To further support our intuition for the flattening of the Phillips curve found by an econometrician ignorant of the data-generating process, we consider a more educated estimation based on the correct non-linear AS equation given by the system of equations (11)–(13) adding to the regression also the square of inflation expectations, the square of output gap and of past inflation rate:

$$\pi_t = a' + b' y_t + \beta \pi^e_{t+1} + \beta' \left(\pi^e_{t+1}\right)^2 + \phi y_t^2 + \gamma \pi^2_{t-1} + \varepsilon_t. \tag{19}$$

The results are presented in Table 3. Although parameters vary across the different values of the standard deviation of the shocks, what is now striking is that the coefficient b' is rather stable when volatility varies. Indeed, the higher volatility is now captured by the quadratic regressors. Note, however, that even regression (19) is misspecified, but it is exactly how it is misspecified that matters for the different values of the coefficient of the output gap estimated through model economies with different volatilities. Any interpretation of structural changes in the slope of the Phillips curve could be misled were the true data-generating process known.

	Benchmark	Case 2σ	Case 5σ	Case 10σ
a'	-0.0003	-0.0001	-0.0004	-0.002
	(-0.004 0.004)	(-0.001 0.0008)	(-0.003, 0.002)	(-0.012 0.008)
y_t	0.062	0.059	0.058	0.059
	(0.034, 0.091)	(0.03, 0.090)	(0.019, 0.010)	(-0.004 0.13)
$E_t\pi_{t+1}$	0.885	0.904	0.960	1.000
	(0.80, 0.96)	(0.80, 1.00)	(0.81, 1.10)	(0.820 1.18)
$E_t\pi^2_{t+1}$	25.13	40.29	20.33	3.47
	(-264.41 309.99)	(-91.68 159.60)	(-16.04, 57.44)	(-8.60 16.06)
y_t^2	-0.785	-0.771	-0.534	-0.42
	(-2.02 0.44)	(-1.41 -0.13)	(-0.87, -0.22)	(-0.69,-0.17)
π^2_{t-1}	-20.09	-33.07	-15.1947	-2.270
	(-266.93, 232.21)	(-133.79, 79.79)	(-44.36, 13.29)	(-12.04, 7.05)
R^2	0.893	0.88	0.823	0.769

Table 3: Estimates of equation (19). Cases 2σ, 5σ, 10σ consider respectively 2, 5, 10 times the standard deviation of the benchmark case. Confidence intervals are reported below each estimate. The coefficients and the standard deviations are the median values of the 5.000 estimations.

7 Conclusion

We have studied the nonlinearities embedded in the standard New Keynesian model. A cubic approximation of the welfare shows that the policymaker should fear more expansions in output and inflation rather than contractions. A second-order approximation of the aggregate-supply equation implies that upward pressures coming from real marginal costs do not necessarily push up inflation. In the optimal policy problem, an overall expansionary bias emerges, implying a relatively higher inflation following both positive or negative mark-up shocks with respect to what would have been implied by the standard linear-quadratic analysis. We use our framework to run a counterfactual experiment on U.S. data to study how optimal policy should have been conducted, comparing the case in which the monetary policymaker was following the linear model to the one in which he followed the non-linear model. Finally, we argue that one of the possible reasons for the flattening of the Phillips curve can be related to an omitted-variable problem for which the econometrician disregards second-order terms from the estimation, whose variation can be responsible for the flattening of the estimated linear relationship between output and inflation.

Our paper suggests that considering nonlinearities can be an important element for appropriately conducting monetary policy, in particular when volatility varies substantially over time. The analysis should be extended to more general frameworks, which could further enhance the importance of nonlinearities.

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A Appendix

In this Appendix we derive the approximations present in the text.

A.1 Derivation of equation (9)

We take a third-order approximation of expression for the index of price dispersion:

$$\Delta_t \equiv \alpha \Delta_{t-1} \left(\frac{\Pi_t}{\Pi}\right)^{\sigma} + (1-\alpha) \left(\frac{1-\alpha\left(\frac{\Pi_t}{\Pi}\right)^{\sigma-1}}{1-\alpha}\right)^{\frac{\sigma}{\sigma-1}}.$$

First, note that

$$\begin{aligned}
\Delta_t - 1 &= \alpha(\Delta_{t-1}-1) + \alpha\left[\sigma\left(\frac{\Pi_t-\Pi}{\Pi}\right) + \frac{1}{2}\sigma(\sigma-1)\left(\frac{\Pi_t-\Pi}{\Pi}\right)^2 + \frac{1}{6}\sigma(\sigma-1)(\sigma-2)\left(\frac{\Pi_t-\Pi}{\Pi}\right)^3\right] + \\
&\quad + \alpha\sigma(\Delta_{t-1}-1)\left(\frac{\Pi_t-\Pi}{\Pi}\right) - \alpha\sigma\left(\frac{\Pi_t-\Pi}{\Pi}\right) - \frac{1}{2}\frac{\sigma\alpha}{(1-\alpha)}(\sigma+\alpha-\sigma\alpha-2)\left(\frac{\Pi_t-\Pi}{\Pi}\right)^2 + \\
&\quad - \frac{1}{6}\frac{\sigma\alpha(\sigma-2)}{(1-\alpha)^2}\left(\sigma+3\alpha-\alpha^2-2\sigma\alpha+\sigma\alpha^2-3\right)\left(\frac{\Pi_t-\Pi}{\Pi}\right)^3 + \mathcal{O}(||\xi||^4)
\end{aligned}$$

which can be written as

$$\begin{aligned}
\hat{\Delta}_t &= \alpha\hat{\Delta}_{t-1} + \alpha\sigma\hat{\Delta}_{t-1}(\pi_t-\pi) + \frac{1}{2}\alpha\sigma(\sigma-1)(\pi_t-\pi)^2 + \frac{1}{6}\alpha\sigma\left(\sigma^2-1\right)(\pi_t-\pi)^3 \\
&\quad - \frac{1}{2}\frac{\sigma\alpha}{(1-\alpha)}(\sigma+\alpha-\sigma\alpha-2)(\pi_t-\pi)^2 - \frac{1}{6}\sigma\frac{\alpha}{(1-\alpha)^2}\left(\sigma^2\alpha^2-2\sigma^2\alpha+\sigma^2+\sigma\alpha+\right. \\
&\quad \left. -2\sigma-\alpha^2+3\alpha\right)(\pi_t-\pi)^3 + \mathcal{O}(||\xi||^4)
\end{aligned}$$

having used

$$\Pi_t = \Pi\left[1+(\pi_t-\pi)+\frac{1}{2}(\pi_t-\pi)^2+\frac{1}{6}(\pi_t-\pi)^3\right] + \mathcal{O}(||\xi||^4).$$

We can simplify the above expression to

$$\begin{aligned}
\hat{\Delta}_t &= \alpha\hat{\Delta}_{t-1} + \alpha\sigma\hat{\Delta}_{t-1}(\pi_t-\pi) + \frac{1}{2}\frac{\sigma\alpha}{1-\alpha}(\pi_t-\pi)^2 + \frac{1}{6}\frac{\sigma\alpha}{1-\alpha}\gamma(\pi_t-\pi)^3 \\
&\quad + \mathcal{O}(||\xi||^4)
\end{aligned}$$

where we have defined

$$\gamma \equiv \frac{(\sigma-1)}{(1-\alpha)} + \sigma - \frac{\alpha}{1-\alpha}.$$

Now note that

$$\begin{aligned}
\hat{\Delta}_t &= \alpha^{t-t_0+1}\hat{\Delta}_{t_0-1} + \alpha\sigma\sum_{s=t_0}^{t}\alpha^{t-s}\,\hat{\Delta}_{s-1}(\pi_s-\pi) + \frac{1}{2}\frac{\sigma\alpha}{(1-\alpha)}\sum_{s=t_0}^{t}\alpha^{t-s}\,(\pi_s-\pi)^2 + \\
&\quad + \frac{1}{6}\frac{\sigma\alpha}{(1-\alpha)}\gamma\sum_{s=t_0}^{t}\alpha^{t-s}(\pi_s-\pi)^3 + \mathcal{O}(||\xi||^4),
\end{aligned}$$

and therefore

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} \hat{\Delta}_t = \frac{\alpha}{(1-\alpha\beta)} \hat{\Delta}_{t_0-1} + \frac{\alpha\sigma}{(1-\alpha\beta)} \sum_{t=t_0}^{\infty} \beta^{t-t_0}\ \hat{\Delta}_{t-1}(\pi_t - \pi) + \frac{1}{2} \frac{\alpha\sigma}{(1-\alpha)(1-\alpha\beta)} \sum_{t=t_0}^{\infty} \beta^{t-t_0}\ (\pi_t - \pi)^2$$
$$+\frac{1}{6} \frac{\alpha\sigma\gamma}{(1-\alpha)(1-\alpha\beta)} \sum_{t=t_0}^{\infty} \beta^{t-t_0} (\pi_t - \pi)^3 + \mathcal{O}(||\xi||^4).$$

Note, moreover, that up to second-order terms, it follows:

$$\hat{\Delta}_t = \alpha^{t-t_0+1} \hat{\Delta}_{t_0-1} + \frac{1}{2} \frac{\alpha}{(1-\alpha)} \sigma \sum_{s=t_0}^{t} \alpha^{t-s}\ (\pi_s - \pi)^2 + \mathcal{O}(||\xi||^3).$$

A.2 Derivation of AS equation (11)

The AS relation can be written exactly as

$$\log \left(\frac{1 - \alpha \left(\frac{\Pi_t}{\Pi}\right)^{\sigma-1}}{1-\alpha} \right) = -(\sigma - 1)(\log K_t - \log F_t), \tag{A.1}$$

where

$$F_t = 1 + \alpha\beta E_t \left\{ F_{t+1} \left(\frac{\Pi_{t+1}}{\Pi} \right)^{\sigma-1} \right\},$$
$$K_t = k_t + \alpha\beta E_t \left\{ K_{t+1} \left(\frac{\Pi_{t+1}}{\Pi} \right)^{\sigma} \right\}.$$

A second-order Taylor series for the left-hand side of (A.1) takes the form

$$\log \left(1 - \frac{\alpha \left(\frac{\Pi_t}{\Pi}\right)^{\sigma-1}}{1-\alpha} \right) = -\frac{\alpha}{1-\alpha}(\sigma - 1) \left\{ (\pi_t - \pi) + \frac{1}{2} \frac{\sigma - 1}{1-\alpha} (\pi_t - \pi)^2 + \mathcal{O}(||\xi||^3). \right\} \tag{A.2}$$

It remains to derive similar second-order approximations for $\log K_t$ and $\log F_t$ on the right-hand side.

The definitions of K_t and F_t imply second-order expansions

$$\hat{F}_t + \frac{1}{2}\hat{F}_t^2 + \mathcal{O}(||\xi||^3) = \alpha\beta E_t \{ \hat{F}_{t+1} + \frac{1}{2}\hat{F}_{t+1}^2 + (\sigma - 1)(\pi_{t+1} - \pi) + \frac{(\sigma-1)^2}{2}(\pi_{t+1} - \pi)^2$$
$$+(\sigma - 1)(\pi_{t+1} - \pi)\hat{F}_{t+1} \} + \mathcal{O}(||\xi||^3), \tag{A.3}$$

$$\hat{K}_t + \frac{1}{2}\hat{K}_t^2 + \mathcal{O}(||\xi||^3) = (1-\alpha\beta) \left(\hat{k}_t + \frac{1}{2}\hat{k}_t^2 \right) + \alpha\beta E_t \{ \hat{K}_{t+1} + \frac{1}{2}\hat{K}_{t+1}^2 + \sigma(\pi_{t+1} - \pi) +$$
$$+\frac{\sigma^2}{2}(\pi_{t+1} - \pi)^2 + \sigma(\pi_{t+1} - \pi)\hat{K}_{t+1} \} + \mathcal{O}(||\xi||^3). \tag{A.4}$$

First, note that, to a first-order approximation,

$$\frac{\alpha}{1-\alpha}(\pi_t - \pi) = (\hat{K}_t - \hat{F}_t) + \mathcal{O}(||\xi||^2) \tag{A.5}$$

$$\begin{aligned}
\hat{F}_t &= \alpha\beta E_t\{\hat{F}_{t+1} + (\sigma-1)(\pi_{t+1}-\pi)\} + \mathcal{O}(||\xi||^2) \\
\hat{K}_t &= (1-\alpha\beta)\hat{k}_t + \alpha\beta E_t\{\hat{K}_{t+1} + \sigma(\pi_{t+1}-\pi)\} + \mathcal{O}(||\xi||^2)
\end{aligned}$$

and therefore

$$\begin{aligned}
(\pi_t - \pi) &= \frac{1-\alpha}{\alpha}((1-\alpha\beta)\hat{k}_t + \alpha\beta E_t\{\hat{K}_{t+1} - \hat{F}_{t+1} + (\pi_{t+1}-\pi)\}) \\
&= \frac{1-\alpha}{\alpha}(1-\alpha\beta)\hat{k}_t + \beta E_t\{\pi_{t+1}-\pi\} + \mathcal{O}(||\xi||^2).
\end{aligned}$$

Note that we can also write

$$\hat{F}_t = (\sigma-1)E_t\sum_{T=t+1}^{\infty}(\alpha\beta)^{T-t}(\pi_T - \pi) + \mathcal{O}(||\xi||^2)$$

and

$$\begin{aligned}
\hat{K}_t = {} & \frac{\alpha}{1-\alpha}E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}(\pi_T-\pi) + \left(\sigma - \frac{1}{1-\alpha}\right)E_t\sum_{T=t+1}^{\infty}(\alpha\beta)^{T-t}E_t\{\pi_T - \pi\} + \\
& +\mathcal{O}(||\xi||^2).
\end{aligned}$$

Take now the difference between (A.3) and (A.4) to obtain

$$\begin{aligned}
\hat{K}_t - \hat{F}_t + \frac{1}{2}(\hat{K}_t - \hat{F}_t)(\hat{F}_t + \hat{K}_t) = {} & (1-\alpha\beta)\left(\hat{k}_t + \frac{1}{2}\hat{k}_t^2\right) + \alpha\beta E_t\{(\hat{K}_{t+1} - \hat{F}_{t+1}) + \\
& \frac{1}{2}(\hat{K}_{t+1} - \hat{F}_{t+1})(\hat{K}_{t+1} + \hat{F}_{t+1}) + \\
& +(\pi_{t+1}-\pi) + \frac{2\sigma-1}{2}(\pi_{t+1}-\pi)^2 - (\sigma-1)(\pi_{t+1}-\pi)\hat{F}_{t+1} \\
& +\sigma(\pi_{t+1}-\pi)\hat{K}_{t+1}\} + \mathcal{O}(||\xi||^3).
\end{aligned}$$

Using (A.1), (A.2) and (A.5), we can write

$$\begin{aligned}
(\pi_t - \pi) + \frac{1}{2}\frac{\sigma-1}{(1-\alpha)}(\pi_t-\pi)^2 + \frac{1}{2}(\pi_t-\pi)Z_t = {} & \frac{1-\alpha}{\alpha}(1-\alpha\beta)\left(\hat{k}_t + \frac{1}{2}\hat{k}_t^2\right) + \beta E_t\{(\pi_{t+1}-\pi) + \\
& \frac{\alpha}{2}\frac{\sigma-1}{(1-\alpha)}(\pi_{t+1}-\pi)^2 + \frac{1}{2}\alpha(\pi_{t+1}-\pi)Z_{t+1} \\
& +(1-\alpha)\frac{2\sigma-1}{2}(\pi_{t+1}-\pi)^2 + \\
& -(1-\alpha)(\sigma-1)(\pi_{t+1}-\pi)\hat{F}_{t+1} \\
& +(1-\alpha)\sigma(\pi_{t+1}-\pi)\hat{K}_{t+1}\} + \mathcal{O}(||\xi||^3),
\end{aligned}$$

where we have defined $Z_t = \hat{F}_t + \hat{K}_t$.

Now note that

$$\begin{aligned}
& \frac{1}{2}(\alpha-1)(\pi_{t+1}-\pi)Z_{t+1} - (1-\alpha)(\sigma-1)(\pi_{t+1}-\pi)\hat{F}_{t+1} + \\
+(1-\alpha)\sigma(\pi_{t+1}-\pi)\hat{K}_{t+1} &= \frac{1}{2}(\alpha-1)(\pi_{t+1}-\pi)(\hat{F}_{t+1}+\hat{K}_{t+1}) - (1-\alpha)(\sigma-1)(\pi_{t+1}-\pi)\hat{F}_{t+1} + \\
+(1-\alpha)\sigma(\pi_{t+1}-\pi)\hat{K}_{t+1} &= -\frac{(1-\alpha)}{2}(2\sigma-1)(\pi_{t+1}-\pi)(\hat{F}_{t+1}-\hat{K}_{t+1}) \\
&= \frac{\alpha}{2}(2\sigma-1)(\pi_{t+1}-\pi)^2 + \mathcal{O}(||\xi||^3).
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
(\pi_t-\pi) + \frac{1}{2}\frac{\sigma-1}{(1-\alpha)}(\pi_t-\pi)^2 + \frac{1}{2}(\pi_t-\pi)Z_t &= \frac{1-\alpha}{\alpha}(1-\alpha\beta)\left(\hat{k}_t + \frac{1}{2}\hat{k}_t^2\right) + \beta E_t\{(\pi_t-\pi) \\
&\quad + \frac{\alpha}{2}\frac{\sigma-1}{(1-\alpha)}(\pi_{t+1}-\pi)^2 + \frac{1}{2}(\pi_{t+1}-\pi)Z_{t+1} \\
&\quad + \frac{2\sigma-1}{2}(\pi_{t+1}-\pi)^2\} + \mathcal{O}(||\xi||^3)
\end{aligned}$$

which can be further rewritten as

$$\begin{aligned}
(\pi_t-\pi) + \frac{1}{2}\frac{\sigma-1}{(1-\alpha)}(\pi_t-\pi)^2 + \frac{1}{2}(\pi_t-\pi)Z_t &= \frac{1-\alpha}{\alpha}(1-\alpha\beta)\left(\hat{k}_t + \frac{1}{2}\hat{k}_t^2\right) + \beta E_t\{(\pi_t-\pi) \\
&\quad + \frac{1}{2}\frac{\sigma-1}{(1-\alpha)}(\pi_{t+1}-\pi)^2 + \frac{1}{2}(\pi_{t+1}-\pi)Z_{t+1} \\
&\quad + \frac{\sigma}{2}(\pi_{t+1}-\pi)^2\} + \mathcal{O}(||\xi||^3).
\end{aligned}$$

Therefore, defining

$$V_t \equiv (\pi_t-\pi) + \frac{1}{2}\left(\sigma + \frac{\sigma-1}{(1-\alpha)}\right)(\pi_t-\pi)^2 + \frac{1}{2}(\pi_t-\pi)Z_t$$

we can write it as

$$V_t = \frac{1-\alpha}{\alpha}(1-\alpha\beta)\left[\hat{k}_t + \frac{1}{2}\hat{k}_t^2\right] + \frac{\sigma}{2}(\pi_t-\pi)^2 + \beta E_t V_{t+1}.$$

Consider now

$$\begin{aligned}
\hat{F}_t &= \alpha\beta E_t\{\hat{F}_{t+1} + (\sigma-1)(\pi_{t+1}-\pi)\} + \mathcal{O}(||\xi||^2) \\
\hat{K}_t &= (1-\alpha\beta)\hat{k}_t + \alpha\beta E_t\{\hat{K}_{t+1} + \sigma(\pi_{t+1}-\pi)\} + \mathcal{O}(||\xi||^2)
\end{aligned}$$

$$\hat{F}_t = (\sigma-1)E_t\sum_{T=t+1}^{\infty}(\alpha\beta)^{T-t}(\pi_T-\pi) + \mathcal{O}(||\xi||^2)$$

and

$$\hat{K}_t = \frac{\alpha}{1-\alpha}E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}(\pi_T-\pi) + \left(\sigma - \frac{1}{1-\alpha}\right)E_t\sum_{T=t+1}^{\infty}(\alpha\beta)^{T-t}E_t\{\pi_T-\pi\} + \mathcal{O}(||\xi||^2).$$

$$
\begin{aligned}
Z_t &= \hat{K}_t + \hat{F}_t \\
&= \frac{\alpha}{1-\alpha} E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}(\pi_T - \pi) + \left(-\frac{1}{1-\alpha} + 2\sigma - 1\right) E_t \sum_{T=t+1}^{\infty} (\alpha\beta)^{T-t} E_t\{\pi_T - \pi\}. \\
&= -\left(-\frac{1}{1-\alpha} + 2\sigma - 1\right)(\pi_t - \pi) + 2(\sigma-1)E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}(\pi_T - \pi) + \mathcal{O}(||\xi||^2).
\end{aligned}
$$

Therefore, we can write

$$
V_t \equiv (\pi_t - \pi) + \frac{1}{2}\left(1 + \sigma\frac{\alpha}{1-\alpha}\right)(\pi_t - \pi)^2 + (\sigma - 1)(\pi_t - \pi)X_t
$$

having defined

$$
X_t = (\pi_t - \pi) + \alpha\beta E_t X_{t+1}.
$$

Now note that

$$
\hat{k}_t = (1+\phi)n_t - \Delta_t + \hat{\mu}_t
$$

and that

$$
\hat{\Delta}_t = \alpha\hat{\Delta}_{t-1} + \frac{1}{2}\frac{\alpha}{(1-\alpha)}\sigma(\pi_t - \pi)^2 + \mathcal{O}(||\xi||^3).
$$

Therefore, we can write

$$
\begin{aligned}
V_t &= \frac{1-\alpha}{\alpha}(1-\alpha\beta)\left[(1+\phi)n_t - \Delta_t + \hat{\mu}_t + \frac{1}{2}((1+\phi)n_t + \hat{\mu}_t)^2\right] + \frac{\sigma}{2}(\pi_t - \pi)^2 + \beta E_t V_{t+1} \\
&= \kappa n_t + u_t + \frac{(1+\phi)\kappa}{2}\left(n_t + \frac{u_t}{\kappa}\right)^2 + (1-\alpha)(\beta\hat{\Delta}_t - \hat{\Delta}_{t-1}) + \beta E_t V_{t+1},
\end{aligned}
$$

where we have defined:

$$
\begin{aligned}
\kappa &= \frac{1-\alpha}{\alpha}(1-\alpha\beta)(1+\phi) \\
u_t &= \frac{\kappa}{(1+\phi)}\hat{\mu}_t.
\end{aligned}
$$

A.3 Derivation of the optimal targeting rule (15)

Optimal policy follows from the minimization of the following loss function

$$
L_{t_o} = E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left\{\frac{1}{2}n_t^2 + \frac{1}{6}(1+\phi)n_t^3 + \frac{1}{2}\frac{\sigma}{\kappa}(\pi_t - \pi)^2 + \frac{\sigma}{\kappa}(1\{-\alpha)\hat{\Delta}_{t-1}(\pi_t - \pi) + \frac{1}{6}\frac{\sigma}{\kappa}\gamma(\pi_t - \pi)^3\right\} \tag{A.6}
$$

under the constraints

$$
\hat{\Delta}_t = \alpha\hat{\Delta}_{t-1} + \frac{1}{2}\frac{\sigma\alpha}{1-\alpha}(\pi_t - \pi)^2; \tag{A.7}
$$

$$
V_t + (1-\alpha)\Delta_{t-1} = \kappa n_t + u_t + \frac{(1+\phi)\kappa}{2}\left(n_t + \frac{u_t}{\kappa}\right)^2 + \beta E_t\{V_{t+1} + (1-\alpha)\Delta_t\}; \tag{A.8}
$$

$$V_t \equiv (\pi_t - \pi) + \frac{1}{2}\left[1 + \sigma\frac{\alpha}{1-\alpha}\right](\pi_t - \pi)^2 + (\sigma - 1)(\pi_t - \pi)X_t; \tag{A.9}$$

$$X_t = (\pi_t - \pi) + \alpha\beta E_t X_{t+1}. \tag{A.10}$$

Using Lagrange multipliers $\varphi_{1,t}, \varphi_{2,t}, \varphi_{3,t}, \varphi_{4,t}$ attached to the constraints (B.18)-(B.21), the first-order conditions are

$$n_t : n_t + \frac{1}{2}(1+\phi)n_t^2 = \kappa\varphi_{2,t} + (1+\phi)\kappa\left(n_t + \frac{u_t}{\kappa}\right)\varphi_{2,t}, \tag{A.11}$$

$$\begin{aligned}(\pi_t - \pi) \quad &: \quad \frac{\sigma}{\kappa}(\pi_t - \pi) + \frac{1}{2}\frac{\sigma}{\kappa}\gamma(\pi_t - \pi)^2 + \frac{\sigma}{\kappa}(1-\alpha)\hat{\Delta}_{t-1} \\ &= \quad \frac{\sigma\alpha}{1-\alpha}(\pi_t - \pi)\varphi_{1,t} + \varphi_{3,t} + \left[1 + \sigma\frac{\alpha}{1-\alpha}\right]\varphi_{3,t}(\pi_t - \pi) \\ &\quad\; +(\sigma - 1)\varphi_{3,t}X_t + \varphi_{4,t},\end{aligned} \tag{A.12}$$

$$\hat{\Delta}_t : \frac{\sigma}{\kappa}(1-\alpha)\beta E_t(\pi_{t+1} - \pi) + \varphi_{1,t} + \beta(1-\alpha)E_t\{\varphi_{2,t+1} - \varphi_{2,t}\}, = \alpha\beta E_t\varphi_{1,t+1}, \tag{A.13}$$

$$V_t : \varphi_{3,t} = \varphi_{2,t-1} - \varphi_{2,t}, \tag{A.14}$$

$$X_t : \varphi_{4,t} - \alpha\varphi_{4,t-1} = (\sigma - 1)(\pi_t - \pi)\varphi_{3,t}. \tag{A.15}$$

Note that up to first-order terms, we can write

$$\begin{aligned}\frac{\sigma}{\kappa}(\pi_t - \pi) &= \varphi_{3,t}, \\ \varphi_{3,t} &= \varphi_{2,t-1} - \varphi_{2,t}, \\ n_t &= \kappa\varphi_{2,t}.\end{aligned}$$

Moreover, note that (A.13) implies, again up to a first-order approximation, that

$$\begin{aligned}\frac{\sigma}{\kappa}(1-\alpha)\beta E_t(\pi_{t+1} - \pi) + \varphi_{1,t} + \beta(1-\alpha)E_t(\varphi_{2,t+1} - \varphi_{2,t}) &= \alpha\beta E_t\varphi_{1,t+1} \\ \varphi_{1,t} &= \alpha\beta E_t\varphi_{1,t+1}\end{aligned}$$

and therefore

$$\varphi_{1,t} = 0.$$

Note also that (A.15) can be written as

$$\varphi_{4,t} - \alpha\varphi_{4,t-1} = (\sigma - 1)\frac{\sigma}{\kappa}(\pi_t - \pi)^2,$$

and comparing it with (B.18) we note that

$$\varphi_{4,t} = \frac{2(1-\alpha)(\sigma - 1)}{\kappa\alpha}\hat{\Delta}_t.$$

We can insert these results into (A.11) to obtain

$$\begin{aligned}
\varphi_{2,t} &= \frac{1}{\kappa}n_t + \frac{1}{2}\frac{1}{\kappa}(1+\phi)n_t^2 - \frac{(1+\phi)}{\kappa}n_t\left(n_t + \frac{u_t}{\kappa}\right), \\
&= \frac{1}{\kappa}n_t - \frac{1}{2}\frac{1}{\kappa}(1+\phi)n_t^2 - \frac{(1+\phi)}{\kappa^2}n_t u_t, \\
&= \frac{1}{\kappa}n_t - \frac{1}{2}\frac{(1+\phi)}{\kappa}\left(n_t + \frac{u_t}{\kappa}\right)^2 + \frac{1}{2}\frac{(1+\phi)}{\kappa}\frac{u_t^2}{\kappa^2}. \qquad (A.16)
\end{aligned}$$

And into (A.12) to get

$$\begin{aligned}
\sigma(\pi_t - \pi) + \frac{1}{2}\sigma\gamma(\pi_t-\pi)^2 + \sigma(1-\alpha)\hat{\Delta}_{t-1} &= \left[1 + \sigma\frac{\alpha}{1-\alpha}\right]\sigma(\pi_t-\pi)^2 - \kappa(\varphi_{2,t} - \varphi_{2,t-1}) \\
&\quad + \sigma(\sigma-1)X_t(\pi_t - \pi) + \frac{2(1-\alpha)(\sigma-1)}{\alpha}\hat{\Delta}_t
\end{aligned}$$

which can be further arranged as

$$\begin{aligned}
\sigma(\pi_t - \pi) + \frac{1}{2}\sigma\gamma(\pi_t-\pi)^2 + \sigma(1-\alpha)\hat{\Delta}_{t-1} &= \left[1 + \sigma\frac{\alpha}{1-\alpha}\right]\sigma(\pi_t-\pi)^2 - \kappa(\varphi_{2,t} - \varphi_{2,t-1}) \\
&\quad + \sigma(\sigma-1)(\pi_t-\pi)^2 + \sigma(\sigma-1)\alpha\beta(\pi_t-\pi)E_tX_{t+1} \\
&\quad + \sigma(\sigma-1)(\pi_t-\pi)^2 + 2(1-\alpha)(\sigma-1)\hat{\Delta}_{t-1},
\end{aligned}$$

and finally as

$$\begin{aligned}
\sigma(\pi_t - \pi) + \frac{1}{2}\sigma\gamma(\pi_t-\pi)^2 &= \alpha\beta\sigma(\sigma-1)(\pi_t-\pi)E_tX_{t+1} + \frac{\alpha + 2\sigma - \alpha\sigma - 1}{1-\alpha}\sigma(\pi_t-\pi)^2 \\
&\quad - \kappa(\varphi_{2,t} - \varphi_{2,t-1}) + (1-\alpha)(\sigma-2)\hat{\Delta}_{t-1}.
\end{aligned}$$

Note that

$$\gamma = \frac{(\sigma-1)}{(1-\alpha)} + \sigma - \frac{\alpha}{1-\alpha},$$

therefore we can simplify the above expression to

$$\begin{aligned}
\sigma(\pi_t - \pi) + \frac{3\alpha + 2\sigma - \alpha\sigma - 1}{\alpha - 1}\frac{\sigma}{2}(\pi_t-\pi)^2 + (1-\alpha)(2-\sigma)\hat{\Delta}_{t-1} &= \sigma\alpha\beta(\sigma-1)(\pi_t-\pi)E_tX_{t+1} + \\
&\quad - \kappa(\varphi_{2,t} - \varphi_{2,t-1}).
\end{aligned}$$

We can insert in the above expression equation (A.16) to finally get

$$\begin{aligned}
&\sigma(\pi_t - \pi) + (n_t - n_{t-1}) - \frac{1}{2}(1+\phi)\left[\left(n_t + \frac{u_t}{\kappa}\right)^2 - \left(n_{t-1} + \frac{u_{t-1}}{\kappa}\right)^2\right] \\
&+ \frac{\sigma}{2}\left(1 - \sigma - \frac{\sigma + 2\alpha}{1-\alpha}\right)(\pi_t - \pi)^2 \\
= \; &\sigma\alpha\beta(\sigma-1)(\pi_t-\pi)E_tX_{t+1} - (2-\sigma)(1-\alpha)\hat{\Delta}_{t-1} - \frac{1}{2}\frac{(1+\phi)}{\kappa^2}(u_t^2 - u_{t-1}^2).
\end{aligned}$$

We can also write it in terms of the output gap noting that

$$y_t = n_t - \hat{\Delta}_t$$

and therefore

$$\begin{aligned}
&\sigma(\pi_t - \pi) + (y_t - y_{t-1}) + \hat{\Delta}_t - \hat{\Delta}_{t-1} - \frac{1}{2}(1+\phi)\left[\left(n_t + \frac{u_t}{\kappa}\right)^2 - \left(n_{t-1} + \frac{u_{t-1}}{\kappa}\right)^2\right] \\
&+\frac{\sigma}{2}\left(1 - \sigma - \frac{\sigma + 2\alpha}{1-\alpha}\right)(\pi_t - \pi)^2 \\
= \;& \sigma\alpha\beta(\sigma - 1)(\pi_t - \pi)E_t X_{t+1} - (2-\sigma)(1-\alpha)\hat{\Delta}_{t-1} - \frac{1}{2}\frac{(1+\phi)}{\kappa^2}(u_t^2 - u_{t-1}^2).
\end{aligned}$$

Using (B.18), we can simplify it to

$$\begin{aligned}
&\sigma(\pi_t - \pi) + (y_t - y_{t-1}) - (1-\alpha)\,\hat{\Delta}_{t-1} + \frac{1}{2}\frac{\sigma\alpha}{1-\alpha}(\pi_t - \pi)^2 \\
&-\frac{1}{2}(1+\phi)\left[\left(y_t + \frac{u_t}{\kappa}\right)^2 - \left(y_{t-1} + \frac{u_{t-1}}{\kappa}\right)^2\right] + \frac{\sigma}{2}\left(1 - \sigma - \frac{\sigma+2\alpha}{1-\alpha}\right)(\pi_t - \pi)^2 \\
= \;& \sigma\alpha\beta(\sigma - 1)(\pi_t - \pi)E_t X_{t+1} - (2-\sigma)(1-\alpha)\hat{\Delta}_{t-1} - \frac{1}{2}\frac{(1+\phi)}{\kappa^2}(u_t^2 - u_{t-1}^2)
\end{aligned}$$

and finally we get

$$\begin{aligned}
&\sigma(\pi_t - \pi) + (y_t - y_{t-1}) = \frac{1}{2}(1+\phi)\left[\left(y_t + \frac{u_t}{\kappa}\right)^2 - \left(y_{t-1} + \frac{u_{t-1}}{\kappa}\right)^2\right] + \\
&+\frac{\sigma}{2}\left(\frac{\sigma+\alpha}{1-\alpha} + \sigma - 1\right)(\pi_t - \pi)^2 + \alpha\beta\sigma(\sigma-1)(\pi_t - \pi)E_t X_{t+1} + (\sigma-1)(1-\alpha)\hat{\Delta}_{t-1} - \\
&-\frac{1}{2}\frac{(1+\phi)}{\kappa^2}(u_t^2 - u_{t-1}^2).
\end{aligned}$$

We can further express it as:

$$\begin{aligned}
\sigma(\pi_t - \pi) + (y_t - y_{t-1}) \;=\; & \frac{(1+\phi)}{2}(y_t^2 - y_{t-1}^2) + \frac{\sigma}{2}\left(\frac{\sigma+\alpha}{1-\alpha} + \sigma - 1\right)(\pi_t - \pi)^2 \\
& +\alpha\beta\sigma(\sigma-1)(\pi_t - \pi)E_t X_{t+1} + \\
& +(\sigma-1)(1-\alpha)\hat{\Delta}_{t-1} + \frac{(1+\phi)}{\kappa}(y_t u_t - y_{t-1}u_{t-1}).
\end{aligned}$$

$$\sigma(\pi_t - \pi) + (y_t - y_{t-1}) = \mathcal{T}_t,$$

which can be rewritten as

$$\mathcal{T}_t = \tau_1(\pi_t - \pi)^2 + \tau_2(\pi_t - \pi)E_t X_{t+1} + \tau_3\hat{\Delta}_{t-1} + \tau_4(y_t^2 - y_{t-1}^2) + \tau_5(y_t u_t - y_{t-1}u_{t-1}),$$

with

$$\begin{aligned}
\tau_1 &= \frac{\sigma}{2}\left(\frac{\sigma+\alpha}{1-\alpha}+\sigma-1\right), \\
\tau_2 &= \alpha\beta\sigma(\sigma-1), \\
\tau_3 &= (\sigma-1)(1-\alpha), \\
\tau_4 &= \frac{(1+\phi)}{2}, \\
\tau_5 &= \frac{(1+\phi)}{\kappa}.
\end{aligned}$$

B Note on filtering mark-up shocks

The aim is to back up and estimate the mark-up shock from our baseline non-linear AS, as well as from the standard linear AS and to compare the implications of the linear model to those of the non-linear (second-order approximated) model.

First, consider the VAR model for the vector of variables $z_t = [y_t, \pi_t]$, output gap and inflation,

$$z_t = Az_{t-1} + \varepsilon_t, \tag{B.17}$$

with Ω being the variance-covariance matrix of the shock ε_t. The output gap is therefore given by

$$y_t = e_2' z_t,$$

where $e_2 = [1;\ 0]$ is a vector that selects the second component of z_t.

Consider now our AS model

$$\hat{\Delta}_t = \alpha\hat{\Delta}_{t-1} + \frac{1}{2}\frac{\sigma\alpha}{1-\alpha}(\pi_t-\pi)^2; \tag{B.18}$$

$$V_t + (1-\alpha)\Delta_{t-1} = \kappa n_t + u_t + \frac{(1+\phi)\kappa}{2}\left(n_t+\frac{u_t}{\kappa}\right)^2 + \beta E_t\{V_{t+1} + (1-\alpha)\Delta_t\}; \tag{B.19}$$

$$V_t \equiv (\pi_t-\pi) + \frac{1}{2}\left[1+\sigma\frac{\alpha}{1-\alpha}\right](\pi_t-\pi)^2 + (\sigma-1)(\pi_t-\pi)X_t; \tag{B.20}$$

$$X_t = (\pi_t-\pi) + \alpha\beta E_t X_{t+1}. \tag{B.21}$$

Note that, given

$$y_t = n_t - \hat{\Delta}_t$$

we can write

$$V_t + (1-\alpha)\hat{\Delta}_{t-1} = \kappa y_t + [\kappa+\beta(1-\alpha)]\hat{\Delta}_t + u_t + \frac{(1+\phi)\kappa}{2}\left(y_t+\frac{u_t}{\kappa}\right)^2 + \beta E_t\{V_{t+1}\}. \tag{B.22}$$

Assume that the mark-up shock follows an AR(1) process, as

$$u_t = \rho u_{t-1} + \xi_t \tag{B.23}$$

where σ_ξ^2 is the variance of ξ_t.

Given the process for output gap reported above, guess the following state-space representation

$$
\begin{aligned}
V_t &= v + v_y' \underset{3x1}{z_t} + z_t' v_{yy} z_t + v_\Delta \hat{\Delta}_{t-1} + v_u u_t + v_{\text{uu}} u_t^2 + z_t' v_{yu} u_t \\
\pi_t - \pi &= \bar{\pi} + \pi_y' z_t + z_t' \pi_{yy} z_t + \pi_\Delta \hat{\Delta}_{t-1} + \pi_u u_t + \pi_{\text{uu}} u_t^2 + z_t' \pi_{\text{yu}} u_t \\
\hat{\Delta}_t &= \Delta_\Delta \hat{\Delta}_{t-1} + z_t' \Delta_{yy} z_t + \Delta_{\text{uu}} u_t^2 + z_t' \Delta_{\text{yu}} u_t \\
X_t &= x_y' z_t + x_u u_t.
\end{aligned}
$$

The next step will be to determine all coefficients and matrices using the method of undetermined coefficients.

Note first that (B.18) implies

$$
\Delta_\Delta \hat{\Delta}_{t-1} + z_t' \Delta_{yy} z_t + \Delta_{\text{uu}} u_t^2 + z_t' \Delta_{\text{yu}} u_t = \alpha \hat{\Delta}_{t-1} + \frac{1}{2} \frac{\sigma \alpha}{1-\alpha} (z_t' \pi_y \pi_y' z_t + \pi_u^2 u_t^2 + 2\pi_u z_t' \pi_y u_t)
$$

and therefore

$$
\begin{aligned}
\Delta_\Delta &= \alpha \\
\Delta_{yy} &= \frac{1}{2} \frac{\sigma \alpha}{1-\alpha} \pi_y \pi_y' \\
\Delta_{\text{uu}} &= \frac{1}{2} \frac{\sigma \alpha}{1-\alpha} \pi_u^2 \\
\Delta_{\text{yu}} &= \frac{\sigma \alpha}{1-\alpha} \pi_u \pi_y
\end{aligned}
$$

Note that Δ_{yy} is a 3 by 3 matrix, while Δ_{yu} is a 3 by 1 vector.

Use (B.22) to obtain

$$
\begin{aligned}
& v + v_y' z_t + z_t' v_{yy} z_t + v_\Delta \hat{\Delta}_{t-1} + v_u u_t + \\
& v_{\text{uu}} u_t^2 + z_t' v_{yu} u_t + (1-\alpha) \Delta_{t-1} \\
= \; & \kappa e_2' z_t + [\kappa + \beta(1-\alpha)](\Delta_\Delta \hat{\Delta}_{t-1} + z_t' \Delta_{yy} z_t + \Delta_{\text{uu}} u_t^2 + \\
& + z_t' \Delta_{\text{yu}} u_t) + u_t + \frac{(1+\phi)\kappa}{2} \left(z_t' e_2 e_2' z_t + \frac{u_t^2}{\kappa^2} + 2 \frac{u_t}{\kappa} z_t' e_2 \right) \\
& + \beta E_t \{ v + v_y' z_{t+1} + z_{t+1}' v_{yy} z_{t+1} + v_\Delta \Delta_t + v_u u_{t+1} + v_{\text{uu}} u_{t+1}^2 + z_{t+1}' v_{yu} u_{t+1} \};
\end{aligned}
$$

and therefore

$$
\begin{aligned}
& v + v_y' z_t + z_t' v_{yy} z_t + v_\Delta \hat{\Delta}_{t-1} + v_u u_t + \\
& v_{\text{uu}} u_t^2 + z_t' v_{yu} u_t + (1-\alpha) \Delta_{t-1} \\
= \; & \kappa e_2' z_t + [\kappa + \beta(1-\alpha) + \beta v_\Delta](\Delta_\Delta \hat{\Delta}_{t-1} + z_t' \Delta_{yy} z_t + \Delta_{\text{uu}} u_t^2 + \\
& + z_t' \Delta_{\text{yu}} u_t) + u_t + \frac{(1+\phi)\kappa}{2} \left(z_t' e_2 e_2' z_t + \frac{u_t^2}{\kappa^2} + 2 \frac{u_t}{\kappa} z_t' e_2 \right) + \\
& + \beta E_t \{ v + v_y' A z_t + z_t' A' v_{yy} A z_t + \varepsilon_{t+1} v_{yy} \varepsilon_{t+1}) + v_u \rho u_t + \\
& v_{\text{uu}} (\rho^2 u_t^2 + \sigma_\xi^2) + z_t' A' v_{\text{yu}} \rho u_t \}.
\end{aligned}
$$

We have the following restrictions:

$$v = \frac{\beta}{1-\beta}(tr[v_{yy}\Omega] + v_{\text{uu}}\sigma_\xi^2)$$

$$v_y' = \kappa e_2' + \beta v_y' A$$

$$v_{yy} = [\kappa + \beta(1-\alpha) + \beta v_\Delta]\Delta_{yy} + \frac{(1+\phi)\kappa}{2} e_2 e_2' + \beta(A' v_{yy} A)$$

$$v_\Delta + (1-\alpha) = [\kappa + \beta(1-\alpha) + \beta v_\Delta]\alpha$$

$$v_u = 1 + \beta v_u \rho$$

$$v_{\text{uu}} = [\kappa + \beta(1-\alpha) + \beta v_\Delta]\Delta_{\text{uu}} + \frac{(1+\phi)}{2\kappa} + \beta v_{\text{uu}}\rho^2$$

$$v_{\text{y}u} = [\kappa + \beta(1-\alpha) + \beta v_\Delta]\Delta_{\text{y}u} + (1+\phi)e_2 + \beta A' v_{\text{y}u}\rho.$$

Moreover,

$$X_t = (\pi_t - \pi) + \alpha\beta E_t X_{t+1}$$

implies that

$$x_y' z_t + x_u u_t = \pi_y' z_t + \pi_u u_t + \alpha\beta(x_y' A z_t + x_u \rho u_t)$$

and therefore we have the following restrictions:

$$x_y' = \pi_y' + \alpha\beta x_y' A$$

$$x_u = \pi_u + \alpha\beta x_u \rho.$$

Finally,

$$\begin{aligned} & v + v_y' z_t + z_t' v_{yy} z_t + v_\Delta \hat{\Delta}_{t-1} + \\ v_u u_t + v_{\text{uu}} u_t^2 + z_t' v_{yu} u_t \quad = \quad & \bar{\pi} + \pi_y' z_t + z_t' \pi_{yy} z_t + \pi_\Delta \Delta_{t-1} + \pi_u u_t + \pi_{\text{uu}} u_t^2 + z_t' \pi_{\text{y}u} u_t \\ & + \frac{1}{2}\left[1 + \sigma\frac{\alpha}{1-\alpha}\right](z_t'\pi_y\pi_y' z_t + 2z_t'\pi_y\pi_u u_t + \pi_u^2 u_t^2) + \\ & + (\sigma - 1)(\pi_y' z_t + \pi_u u_t)'(x_y' z_t + x_u u_t). \end{aligned}$$

Therefore,

$$\bar{\pi} = v$$

$$\pi_y' = v_y'$$

$$\pi_\Delta = v_\Delta$$

$$\pi_u = v_u$$

$$\pi_{yy} = v_{yy} - \frac{1}{2}\left[1 + \sigma\frac{\alpha}{1-\alpha}\right]\pi_y\pi_y' - (\sigma-1)\pi_y x_y'$$

$$\pi_{\text{uu}} = v_{\text{uu}} - \frac{1}{2}\left[1 + \sigma\frac{\alpha}{1-\alpha}\right]\pi_u^2 - (\sigma-1)\pi_u x_u$$

$$\pi_{\mathrm{yu}} = v_{\mathrm{yu}} - \left[1 + \sigma \frac{\alpha}{1-\alpha}\right] \pi_y \pi_u - (\sigma - 1)(\pi_y x_u + \pi_u x_y).$$

To sum up, the list of the parameters and matrices is given by:
$1 : \Delta_{\Delta} = \alpha$
$2 : v_y = [I - \beta A']^{-1} k e_2$
$3 : v_u = \frac{1}{1-\beta\rho}$
$4 : v_{\Delta} = \frac{[\kappa + \beta(1-\alpha)]\alpha - (1-\alpha)}{1-\beta\alpha}$
$5 : \pi'_y = v'_y$
$6 : \pi_{\Delta} = v_{\Delta}$
$7 : \pi_u = v_u$
$8 : \Delta_{yy} = \frac{1}{2}\frac{\sigma\alpha}{1-\alpha}\pi_y \pi'_y$
$9 : \Delta_{\mathrm{uu}} = \frac{1}{2}\frac{\sigma\alpha}{1-\alpha}\pi_u^2$
$10 : \Delta_{\mathrm{yu}} = \frac{\sigma\alpha}{1-\alpha}\pi_u \pi_y$
$11 : v_{yy} = [\kappa + \beta(1-\alpha) + \beta v_{\Delta}]\Delta_{yy} + \frac{(1+\phi)\kappa}{2} e_2 e'_2 + \beta(A' v_{yy} A)$
$12 : v_{\mathrm{uu}} = \frac{[\kappa + \beta(1-\alpha) + \beta v_{\Delta}]\Delta_{\mathrm{uu}} + \frac{(1+\phi)}{2\kappa}}{1-\beta\rho^2}$
$13 : v_{\mathrm{yu}} = [I - \beta A' \rho]^{-1} \left([\kappa + \beta(1-\alpha) + \beta v_{\Delta}]\Delta_{\mathrm{yu}} + (1+\phi) e_2\right)$
$14 : x'_y = \pi'_y + \alpha\beta x'_y A \Longrightarrow x_y = \pi_y + \alpha\beta A' x_y \Longrightarrow x_y = [I - \alpha\beta A']^{-1} \pi_y$
$15 : x_u = \pi_u + \alpha\beta x_u \rho \Longrightarrow x_u = \frac{\pi_u}{1-\alpha\beta\rho}$
$16 : \pi_{yy} = v_{yy} - \frac{1}{2}\left[1 + \sigma\frac{\alpha}{1-\alpha}\right]\pi_y \pi'_y - (\sigma - 1)\pi_y x'_y$
$17 : \pi_{\mathrm{uu}} = v_{\mathrm{uu}} - \frac{1}{2}\left[1 + \sigma\frac{\alpha}{1-\alpha}\right]\pi_u^2 - (\sigma - 1)\pi_u x_u$
$18 : \pi_{\mathrm{yu}} = v_{\mathrm{yu}} - \left[1 + \sigma\frac{\alpha}{1-\alpha}\right]\pi_y \pi_u - (\sigma - 1)(\pi_y x_u + \pi_u x_y)$
$19 : v = \frac{\beta}{1-\beta}(tr[v_{yy}\Omega] + v_{\mathrm{uu}}\sigma_{\xi}^2)$
$20 : \bar{\pi} = v$

Note that all the parameters and matrices are convolutions of the structural parameters of the model, that is, of α, ϕ, σ,β and of the parameters of the markup process, σ_{ξ}^2 and ρ, which are respectively the variance of the innovations of the mark-up shock and its autoregressive component.

Now, assigning a value to all parameters and using the data path of inflation and output gap, it is possible to use the AS equation to back up a path for the mark-up shock as described below. In particular, the parameter β is calibrated equal to 0.99, while the remaining parameters will be estimated with the following procedure.

1. Estimate (B.17) to obtain A, Ω, using data from 1995q1:2019q3 downloaded from FRED database.[8]

2. Guess ρ and σ_{ξ}^2 and a value for a, ϕ, σ.[9]

3. Compute all the coefficients of the non-linear solution derived above.

[8] The output gap is contructuted as the difference between the log of the quarterly series of Real Gross Domestic Product (GDPC1) and the log of the quarterly series of the Potential Output (GDPPOT). The inflation rate is instead constructed as the log difference between the PCE price index excluding food and energy (PCEPILFE). In the VAR analysis we consider demeaned data.

[9] The algorithm uses the function *csmiwel* to minimize the distance between theoretical and data moments.

4. First, as an initial guess of the mark-up process in the non-linear AS, use the values of u_t filtered from the linear AS, that is, use

$$\pi_t - \pi = \pi_y' z_t + \pi_u u_t \tag{B.24}$$

with $\pi_y' = [I - \beta A']^{-1} k e_2$ and $\pi_u = \frac{1}{1-\beta\rho}$ and where π is set to zero since we use demeaned data. Estimate ρ and σ_ξ^2.

5. Given the new values of ρ and σ_ξ^2 compute data statistics on mean of inflation, $E\pi_t$, variance of inflation, $var(\pi_t)$, and covariance between inflation and output, $\mathrm{cov}(\pi_t, y_t)$, and find parameters (σ, ϕ, α) to minimize the distance between data and model statistics. Model statistics are computed using the non-linear AS as

$$E\pi_t = \pi + \bar{\pi} + tr[\pi_{yy} E z_t z_t'] + \pi_\Delta E\Delta + \pi_{\mathrm{uu}} \frac{\sigma_\xi^2}{1-\rho^2}$$

with

$$E\hat{\Delta} = \frac{tr\left[\Delta_{yy} E z_t' z_t\right] + \Delta_{\mathrm{uu}} \frac{\sigma_\xi^2}{1-\rho^2}}{1 - \Delta_\Delta}$$

and

$$var(\pi_t) = tr[E(z_t z_t')\pi_y \pi_y'] + \pi_u^2 \frac{\sigma_\xi^2}{1-\rho^2}$$

$$cov(\pi_t, y_t) = tr[E(z_t z_t') e_2 \pi_y']$$

with

$$E\left(zz'\right) = (I - A)^{-1}\, \Omega \left((I - A)^{-1}\right)'.$$

6. Given $(\rho, \sigma_\xi^2, \sigma, \phi, \alpha)$ estimated above, now use:

$$\pi_t - \pi = \bar{\pi} + \pi_y' z_t + z_t' \pi_{yy} z_t + \pi_\Delta \hat{\Delta}_{t-1} + \pi_u u_t + \pi_{\mathrm{uu}} u_t^2 + z_t' \pi_{\mathrm{yu}} u_t$$

and

$$\hat{\Delta}_t = \Delta_\Delta \hat{\Delta}_{t-1} + z_t' \Delta_{yy} z_t + \Delta_{\mathrm{uu}} u_t^2 + z_t' \Delta_{\mathrm{yu}} u_t$$

with non-demeaned data on π_t and y_t and with a value for target inflation $\pi = 0.005$ to back up $\{u_t\}$ and assuming $\Delta_{t_0-1} = 0$.

7. Estimate

$$u_t = \rho u_{t-1} + \xi_t \tag{B.25}$$

to obtain new values for ρ and σ_ξ^2.

8. Repeat 2-7 until convergence on ρ and σ_ξ^2 is obtained and until the parameters σ, ϕ, α, are those that minimize the distance between theoretical and empirical moments.[10]

[10] The algorithm used to minimize the distance between theoretical moments and empirical moments is the matlab function csminwell.

9. Finally, with the estimated parameters σ, ϕ, α, use again the first order solution (B.24) to back the mark-up filter and estimate again the AR(1) process as in (B.25) to obtain the convergence of the estimates of the parameters ρ and σ_ξ^2.

Results: The structural parameters estimated using steps 1-8 of the procedure are: $\sigma = 4.8$, $\phi = 0.2$ and $\alpha = 0.904$. The estimated ρ and σ_ξ^2 for the non-linear AS are 0.9430 and 0.0111/100, respectively. The parameters ρ and σ_ξ^2 are instead estimated equal to 0.920 and 0185/100 when using the linear AS equation as in step 9.