Inflation Stabilization and Welfare: The Case of a Distorted Steady State*

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Abstract

This paper considers the appropriate stabilization objectives for monetary policy in a microfounded model with staggered price-setting. Rotemberg and Woodford (1997) and Woodford (2002) have shown that under certain conditions, a local approximation to the expected utility of the representative household in a model of this kind is related inversely to the expected discounted value of a conventional quadratic loss function, in which each period's loss is a weighted average of squared deviations of inflation and an output gap measure from their optimal values (zero). However, those derivations rely on an assumption of the existence of an output or employment subsidy that offsets the distortion due to the market power of monopolistically-competitive price-setters, so that the steady state under a zero-inflation policy involves an efficient level of output. Here we show how to dispense with this unappealing assumption, so that a valid linear-quadratic approximation to the optimal policy problem is possible even when the steady state is distorted to an arbitrary extent (allowing for tax distortions as well as market power), and when, as a consequence, it is necessary to take account of the effects of stabilization policy on the average level of output.

We again obtain a welfare-theoretic loss function that involves both inflation and an appropriately defined output gap, though the degree of distortion of the steady state affects both the weights on the two stabilization objectives and the definition of the welfare-relevant output gap. In the light of these results, we reconsider the conditions under which complete price stability is optimal, and find that they are more restrictive in the case of a distorted steady state. We also consider the conditions under which pure randomization of monetary policy can be welfare-improving, and find that this is possible in the case of a sufficiently distorted steady state, though the parameter values required are probably not empirically realistic.

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According to a common conception of the goals of monetary stabilization policy, it is appropriate for the monetary authority to aim to stabilize both some measure of inflation and some measure of real activity relative to potential. This is often represented by supposing that the authority should seek to minimize the expected discounted value of a quadratic loss function, in which each period's loss consists of a weighted average of the square of the inflation rate and the square of the "output gap." It is furthermore typically argued that the two stabilization goals are not fully compatible with one another, owing to the occurrence of "cost-push shocks," which prevent a zero output gap from being consistent with zero inflation. The problem of finding an optimal tradeoff between the two goals is then non-trivial.[1]

This familiar framework raises a number of questions, however. Most obvious is the question of how to define the "output gap" that policy should seek to stabilize. Should this be understood to mean output relative to some smooth trend, or should the target output level vary in response to real disturbances of various sorts? A closely related question is the definition of the "cost-push shocks": how should these be identified in practice, and how often do disturbances of this kind actually occur? And even supposing that we know how to identify the output gap and the cost-push disturbances, what relative weight should be placed on output-gap stabilization as opposed to inflation stabilization?

Here we propose to answer such questions on welfare-theoretic grounds. The ultimate aim of monetary policy, in our view, should be the maximization of the expected utility of households. We show, however (following a method introduced by Rotemberg and Woodford, 1997, and further expounded in Woodford, 2002; 2003, chap. 6), that it is possible to derive a quadratic approximation to the expected utility of the representative household that takes the form of a discounted quadratic loss function of the kind assumed in the traditional literature on monetary policy evaluation. In the case that the exogenous disturbances are sufficiently small in amplitude, the best policy (in terms of expected utility) will also be the one that minimizes the discounted quadratic loss function. We thus obtain precise answers to the question of what terms should appear in a quadratic loss function, and with which relative weights, that depend on the specification of one's model of the monetary transmission mechanism.[2]

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[1]See, *e.g.,* Walsh (2003, chap. xx) for a number of analyses in this vein.

[2]For examples of the way in which alternative model specifications lead to alternative welfare-theoretic loss functions, see Woodford (2003, chap. 6) and Giannoni and Woodford (2003).

An important limitation of the method introduced by Rotemberg and Woodford (1997) is that it requires that the zero-inflation steady state of one's model involve an efficient level of output.[3] (They imagine a model in which this is true by assuming the existence of an output subsidy that offsets the distortion resulting from the market power of monopolistically competitive suppliers, though this is obviously not literally true in actual economies.) For if one were instead to consider the more realistic case of an economy in which steady-state output is inefficiently low, one would find that expected utility would depend on the expected level of output. An estimate of expected utility that is accurate to second order would then require a solution for output (or at any rate, for the expected discounted level of output) that is accurate to second order in the amplitude of the exogenous disturbances. A log-linear approximation to the structural equations of one's model will then not suffice to allow one to determine the evolution of output under one policy or another to a sufficient degree of accuracy. As a consequence, a linear-quadratic methodology — in which a linear policy rule is derived so as to minimize a quadratic approximation to the true welfare objective subject to linear constraints that are first-order approximations to the true structural equations — will not generally yield a correct linear approximation to the optimal policy rule.[4]

Here we show how the method of Rotemberg and Woodford can be extended to deal with the case in which the steady-state level of output is inefficient (owing to the existence of distorting taxes on sales revenues or labor income, in addition to the distortions created by market power). Our approach involves computation of a second-order approximation to the model structural relations (specifically, to the aggregate-supply relation in the present application), and using this to solve for the expected discounted value of output as a function of purely quadratic terms. This solution can then be used to substitute for the terms proportional to expected discounted output in the quadratic approximation to expected utility. In this way, we obtain an approximation to expected utility — that holds regardless of the policy contemplated (as long as it involves inflation that is not too extreme) — and that is

[3]Strictly speaking, it is not essential to the method that zero be the inflation rate that leads to the efficient level of output; it is only necessary that there be *some* such steady state, and that the policies that one intends to compare all be close enough to being consistent with that steady state.

[4]See Woodford (2003, chap. 6) and Benigno and Woodford (2003b) for discussion of the conditions required for validity of an LQ approach.

purely quadratic, in the sense of lacking any linear terms. This alternative quadratic loss function can then be evaluated to second order using an approximate solution for the endogenous variables of one's model that is accurate only to first order. One is then able to compute a linear approximation to optimal policy using a simple linear-quadratic methodology.

Our proposal to substitute purely quadratic terms for the discounted linear terms in the Taylor approximation to expected utility builds upon an idea of Sutherland (2002), who showed how it was possible to take account of the effects of macroeconomic volatility on the average levels of variables in welfare calculations for a model with Calvo pricing like the baseline model considered here. Sutherland's crucial insight was that it is not necessary to compute a complete second-order solution for the evolution of the endogenous variables under each of the policies that one wishes to consider in order to evaluate the discounted linear terms needed for the welfare calculation. Sutherland's approach, however, still requires that one restrict attention to a particular parametric family of policy rules before computing the second-order approximations that are used to substitute for the discounted linear terms in the welfare criterion. Instead, we show that one can substitute out the linear terms using *only* a second-order approximation to the structural equations; one thus obtains a welfare criterion that applies to arbitrary policies.[5]

An alternative way of attaining a welfare measure that is accurate to second order even in the case of a distorted steady state, that has recently become popular, is to solve for a second-order approximation to the complete evolution of the endogenous variables under any given policy rule, and then use this solution to evaluate a quadratic approximation to expected utility (*e.g.,* Kim *et al.,* 2002). However, the requirement that a system of quadratic expectational difference equations be solved for each policy rule that is contemplated is much more computationally demanding than the implementation of our LQ methodology. For we are required to consider the second-order approximation to our structural equations only *once* — when deriving the appropriate quadratic loss function, a calculation undertaken in this paper — after which the evaluation of individual policies requires only that one solve a system

[5]It might seem fortuitous that we are able to do this in the present case, but Benigno and Woodford (2003b) shows that substitutions of this kind can be used quite generally to obtain a purely quadratic loss function. A further application of the method is presented in Benigno and Woodford (2003a).

of linear equations. In addition, the method illustrated by Kim *et al.* requires that one restrict one's attention to a particular parametric family of policy rules, since the system of equations that is solved to second order must include a specification of the policy rule. Our method, by contrast, allows us to determine what variables it is desirable for policy to depend on without having to prejudge that issue.

Yet another approach that allows a correct calculation of a linear approximation to the optimal policy rule even in the case of a distorted steady state is to compute first-order conditions that characterize optimal policy in the exact model (*i.e.,* without approximating either the welfare measure or the structural equations), and then log-linearize these optimality conditions in order to obtain an approximate characterization of optimal policy (*e.g.,* King and Wolman, 1999; Khan *et al.,* 2003). A disadvantage of this approach is that it is *only* suitable for computing the optimal policy; our quadratic approximate welfare measure also yields a correct ranking of alternative sub-optimal policy rules, as long as disturbances are small enough, and the policies under comparison all involve low inflation. Furthermore, our LQ approach makes it straightforward to consider whether the *second-order conditions* for a policy to be a local optimum are satisfied, and not just the first-order conditions that are typically considered in the literature on "Ramsey policy", as we show in section 3.1 below. Under conditions where the second-order conditions are satisfied, our approach and the one used by Khan *et al.* yield identical approximate linear characterizations of optimal policy; but we believe that the LQ approach provides useful insight into the aspects of the policy problem that are responsible for the conclusions obtained. We illustrate this in sections 3.2 and 3.3 by providing an analytical derivation of results with the same qualitative features as the numerical results reported by Khan *et al.* for a related model.

1 Monetary Stabilization Policy: Welfare-Theoretic Foundations

Here we describe our assumptions about the economic environment and pose the optimization problem that a monetary stabilization policy is intended to solve. The approximation method that we use to characterize the solution to this problem is then presented in the following section. Further details of the derivation of the structural equations of our model of nominal price rigidity can be found in Woodford (2003, chapter 3).

1.1 Objective and Constraints

The goal of policy is assumed to be the maximization of the level of expected utility of a representative household. In our model, each household seeks to maximize

$$U_{t_0} \equiv E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\tilde{u}(C_t;\xi_t) - \int_0^1 \tilde{v}(H_t(j);\xi_t)dj \right], \tag{1.1}$$

where C_t is a Dixit-Stiglitz aggregate of consumption of each of a continuum of differentiated goods,

$$C_t \equiv \left[\int_0^1 c_t(i)^{\frac{\theta}{\theta-1}} di \right]^{\frac{\theta-1}{\theta}}, \tag{1.2}$$

with an elasticity of substitution equal to $\theta > 1$, and $H_t(j)$ is the quantity supplied of labor of type j. Each differentiated good is supplied by a single monopolistically competitive producer. There are assumed to be many goods in each of an infinite number of "industries"; the goods in each industry j are produced using a type of labor that is specific to that industry, and also change their prices at the same time. The representative household supplies all types of labor as well as consuming all types of goods.[6] To simplify the algebraic form of our results, in our main exposition we shall restrict attention to the case of isoelastic functional forms,

$$\tilde{u}(C_t;\xi_t) \equiv \frac{C_t^{1-\tilde{\sigma}^{-1}} \bar{C}_t^{\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}}, \tag{1.3}$$

$$\tilde{v}(H_t;\xi_t) \equiv \frac{\lambda}{1+\nu} H_t^{1+\nu} \bar{H}_t^{-\nu}, \tag{1.4}$$

where $\tilde{\sigma}, \nu > 0$, and $\{\bar{C}_t, \bar{H}_t\}$ are bounded exogenous disturbance processes. (We use the notation ξ_t to refer to the complete vector of exogenous disturbances, including $\bar{C}_t$ and $\bar{H}_t$.) The extension of our results to the case of more general preferences is taken up in section xx.

[6] We might alternatively assume specialization across households in the type of labor supplied; in the presence of perfect sharing of labor income risk across households, household decisions regarding consumption and labor supply would all be as assumed here.

We assume a common technology for the production of all goods, in which (industry-specific) labor is the only variable input,

$$y_t(i) = A_t f(h_t(i)) = A_t h_t(i)^{1/\phi},$$

where A_t is an exogenously varying technology factor, and $\phi > 1$. (Again, more general production functions are considered in section xx.) Inverting the production function to write the demand for each type of labor as a function of the quantities produced of the various differentiated goods, and using the identity

$$Y_t = C_t + G_t$$

to substitute for C_t, where G_t is exogenous government demand for the composite good, we can write the utility of the representative household as a function of the expected production plan $\{y_t(i)\}$.[7]

The utility of the representative household (our welfare measure) can be expressed as a function of equilibrium production,

$$U_{t_0} \equiv E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[u(Y_t; \xi_t) - \int_0^1 v(y_t^j; \xi_t) dj \right], \tag{1.5}$$

where

$$u(Y_t; \xi_t) \equiv \tilde{u}(Y_t - G_t; \xi_t),$$
$$v(y_t^j; \xi_t) \equiv \tilde{v}(f^{-1}(y_t^j / A_t); \xi_t).$$

In this last expression we make use of the fact that the quantity produced of each good in industry j will be the same, and hence can be denoted y_t^j; and that the quantity of labor hired by each of these firms will also be the same, so that the total demand for labor of type j is proportional to the demand of any one of these firms.

We can furthermore express the relative quantities demanded of the differentiated goods each period as a function of their relative prices. This allows us to write the utility flow to the representative household in the form $U(Y_t, \Delta_t; \xi_t)$, where

$$\Delta_t \equiv \int_0^1 \left(\frac{p_t(i)}{P_t} \right)^{-\theta(1+\omega)} di \geq 1 \tag{1.6}$$

[7] The government is assumed to need to obtain an exogenously given quantity of the Dixit-Stiglitz aggregate each period, and to obtain this in a cost-minimizing fashion. Hence the government allocates its purchases across the suppliers of differentiated goods in the same proportion as do households, and the index of aggregate demand Y_t is the same function of the individual quantities $\{y_t(i)\}$ as C_t is of the individual quantities consumed $\{c_t(i)\}$, defined in (1.2).

is a measure of price dispersion at date t, in which P_t is the Dixit-Stiglitz price index

$$P_t \equiv \left[\int_0^1 p_t(i)^{1-\theta} di\right]^{\frac{1}{1-\theta}}, \tag{1.7}$$

and the vector ξ_t now includes the exogenous disturbances G_t and A_t as well as the preference shocks. Hence we can write our objective (1.5) as

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} U(Y_t, \Delta_t; \xi_t). \tag{1.8}$$

The producers in each industry fix the prices of their goods in monetary units for a random interval of time, as in the model of staggered pricing introduced by Calvo (1983). We let $0 \leq \alpha < 1$ be the fraction of prices that remain unchanged in any period. A supplier that changes its price in period t chooses its new price $p_t(i)$ to maximize

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi(p_t(i), p_T^j, P_T; Y_T, \xi_T) \right\}, \tag{1.9}$$

where $Q_{t,T}$ is the stochastic discount factor by which financial markets discount random nominal income in period T to determine the nominal value of a claim to such income in period t, and α^{T-t} is the probability that a price chosen in period t will not have been revised by period T. In equilibrium, this discount factor is given by

$$Q_{t,T} = \beta^{T-t} \frac{\tilde{u}_c(C_T; \xi_T)}{\tilde{u}_c(C_t; \xi_t)} \frac{P_t}{P_T}. \tag{1.10}$$

The function

$$\Pi(p, p^j, P; Y, \xi) \equiv (1-\tau) p Y (p/P)^{-\theta} - \mu^w \frac{\tilde{v}_h(f^{-1}(Y(p^j/P)^{-\theta}/A); \xi)}{\tilde{u}_c(Y - G; \xi)} P \cdot f^{-1}(Y(p/P)^{-\theta}/A) \tag{1.11}$$

indicates the after-tax nominal profits of a supplier with price p, in an industry with common price p^j, when the aggregate price index is equal to P and aggregate demand is equal to Y. Here τ_t is the proportional tax on sales revenues in period t; we treat $\{\tau_t\}$ as an exogenous disturbance process, taken as given by the monetary policymaker.[8] We assume that τ_t fluctuates over a small interval around a non-zero

[8]The extension to the case in which the tax rate is also chosen optimally in response to other shocks is treated in Benigno and Woodford (2003a).

steady-state level $\bar{\tau}$; this is another of the possible reasons for inefficiency of the steady-state level of output that we consider.[9] Profits are equal to after-tax sales revenues net of the wage bill, and the real wage demanded for labor of type j is assumed to be given by

$$w_t(j) = \mu_t^w \frac{\tilde{v}_h(H_t(j);\xi)}{\tilde{u}_c(C_t;\xi_t)}, \tag{1.12}$$

where $\mu_t^w \geq 1$ is an exogenous markup factor in the labor market (allowed to vary over time, but assumed common to all labor markets),[10] and firms are assumed to be wage-takers. We allow for exogenous variations in both the tax rate and the wage markup in order to include the possibility of "pure cost-push shocks" that affect equilibrium pricing behavior while implying no change in the efficient allocation of resources.[11] The disturbances τ_t and μ_t^w are also included as elements of the vector ξ_t.

Each of the suppliers that revise their prices in period t choose the same new price p_t^*, that maximizes (1.9). Note that supplier i's profits are a concave function of the quantity sold $y_t(i)$, since revenues are proportional to $y_t^{\frac{\theta-1}{\theta}}(i)$ and hence concave in $y_t(i)$, while costs are convex in $y_t(i)$. Moreover, since $y_t(i)$ is proportional to $p_t(i)^{-\theta}$, the profit function is also concave in $p_t(i)^{-\theta}$. The first-order condition for the optimal choice of the price $p_t(i)$ is the same as the one with respect to $p_t(i)^{-\theta}$; hence the first-order condition with respect to $p_t(i)$,

$$E_t\left\{\sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi_1(p_t(i), p_T^j, P_T; Y_T, \xi_T)\right\} = 0,$$

is both necessary and sufficient for an optimum. The equilibrium choice p_t^* (which is the same for each firm in industry j) is the solution to the equation obtained by substituting $p_t(i) = p_t^j = p_t^*$ into the above.

[9]Other types of distorting taxes would have similar consequences, since it is the overall size of the steady-state inefficiency wedge that is of greatest importance for our analysis, as we show below. To economize on notation, we assume that the only distorting tax is of this particular kind.

[10]In the case that we assume that $\mu_t^w = 1$ at all times, our model is one in which both households and firms are wage-takers, or there is efficient contracting between them.

[11]We show below, however, that these two disturbances are not, in general, the only reasons for the existence of a "cost-push" term in our aggregate-supply relation, in the sense of a term that creates a tension between the goals of inflation stabilization and output-gap stabilization.

Under our assumed isoelastic functional forms, the optimal choice has a closed-form solution

$$\frac{p_t^*}{P_t} = \left(\frac{K_t}{F_t}\right)^{\frac{1}{1+\omega\theta}}, \tag{1.13}$$

where $\omega \equiv \phi(1+\nu) - 1 > 0$ is the elasticity of real marginal cost in an industry with respect to industry output, and F_t and K_t are functions of current aggregate output Y_t, the current exogenous state ξ_t, and the expected future evolution of inflation, output, and disturbances, defined by

$$F_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} (1-\tau_T) f(Y_T; \xi_T) \left(\frac{P_T}{P_t}\right)^{\theta-1}, \tag{1.14}$$

$$K_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} k(Y_T; \xi_T) \left(\frac{P_T}{P_t}\right)^{\theta(1+\omega)}, \tag{1.15}$$

in which expressions

$$f(Y;\xi) \equiv u_y(Y;\xi)Y, \tag{1.16}$$

$$k(Y;\xi) \equiv \frac{\theta}{\theta-1}\mu^w v_y(Y;\xi)Y. \tag{1.17}$$

The price index then evolves according to a law of motion

$$P_t = \left[(1-\alpha)p_t^{*1-\theta} + \alpha P_{t-1}^{1-\theta}\right]^{\frac{1}{1-\theta}}, \tag{1.18}$$

as a consequence of (1.7). Substitution of (1.13) into (1.18) implies that equilibrium inflation in any period is given by

$$\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha} = \left(\frac{F_t}{K_t}\right)^{\frac{\theta-1}{1+\omega\theta}}, \tag{1.19}$$

where $\Pi_t \equiv P_t/P_{t-1}$. This defines a short-run aggregate supply relation between inflation and output, given the current disturbances ξ_t, and expectations regarding future inflation, output, and disturbances. This is the only relevant constraint on the monetary authority's ability to simultaneously stabilize inflation and output in our model.

Because the relative prices of the industries that do not change their prices in period t remain the same, we can also use (1.18) to derive a law of motion of the form

$$\Delta_t = h(\Delta_{t-1}, \Pi_t) \tag{1.20}$$

for the dispersion measure defined in (1.6), where

$$h(\Delta, \Pi) \equiv \alpha \Delta \Pi^{\theta(1+\omega)} + (1-\alpha) \left(\frac{1 - \alpha \Pi^{\theta - 1}}{1-\alpha} \right)^{-\frac{\theta(1+\omega)}{1-\theta}}.$$

This is the source in our model of welfare losses from inflation or deflation.

We assume the existence of a lump-sum source of government revenue (in addition to the fixed tax rate τ), and assume that the fiscal authority ensures intertemporal government solvency regardless of what monetary policy may be chosen by the monetary authority.[12] This allows us to abstract from the fiscal consequences of alternative monetary policies in our consideration of optimal monetary stabilization policy, as is common in the literature on monetary policy rules. An extension of our analysis to the case in which only distorting taxes exist is presented in Benigno and Woodford (2003a).

Finally, we abstract here from any monetary frictions that would account for a demand for central-bank liabilities that earn a substandard rate of return; we nonetheless assume that the central bank can control the riskless short-term nominal interest rate i_t,[13] which is in turn related to other financial asset prices through the arbitrage relation

$$1 + i_t = [E_t Q_{t,t+1}]^{-1}.$$

We shall assume that the zero lower bound on nominal interest rates never binds under the optimal policies considered below,[14] so that we need not introduce any additional constraint on the possible paths of output and prices associated with a need for the chosen evolution of prices to be consistent with a non-negative nominal interest rate. We also note that the ability of the central bank to control i_t in each period gives it one degree of freedom each period (in each possible state of the world) with which to determine equilibrium outcomes. Because of the existence of the aggregate-supply relation (1.19) as a necessary constraint on the joint evolution

[12] Thus we here assume that fiscal policy is "Ricardian," in the terminology of Woodford (2001). A non-Ricardian fiscal policy would imply the existence of an additional constraint on the set of equilibria that could be achieved through monetary policy. The consequences of such a constraint for the character of optimal monetary policy will be considered elsewhere.

[13] For discussion of how this is possible even in a "cashless" economy of the kind assumed here, see Woodford (2003, chapter 2).

[14] This can be shown to be true in the case of small enough disturbances, given that the nominal interest rate is equal to $\bar{r} = \beta^{-1} - 1 > 0$ under the optimal policy in the absence of disturbances.

of inflation and output, there is exactly one degree of freedom to be determined each period, in order to determine particular stochastic processes $\{\Pi_t, Y_t\}$ from among the set of possible rational-expectations equilibria.[15] Hence we shall suppose that the monetary authority can choose from among the possible processes $\{\Pi_t, Y_t\}$ that constitute rational-expectations equilibria, and consider which equilibrium it is optimal to bring about; the detail that policy is implemented through the control of a short-term nominal interest rate will not actually matter to our calculations.

1.2 Optimal Policy from a "Timeless Perspective"

Under the standard (Ramsey) approach to the characterization of an optimal policy commitment, one chooses among state-contingent paths $\{\Pi_t, Y_t, \Delta_t\}$ from some initial date t_0 onward that satisfy (1.19) and (1.20) for each $t \geq t_0$,[16] given initial price dispersion Δ_{t_0-1}, so as to maximize (1.8). Such a t_0-optimal plan requires commitment, insofar as the corresponding $t-$optimal plan for some later date t, given the condition Δ_{t-1} obtaining at that date, will not involve a continuation of the t_0-optimal plan. This failure of time consistency occurs because the constraints on what can be achieved at date t_0, consistent with the existence of a rational-expectations equilibrium, depend on the expected paths of inflation and output at later dates; but in the absence of a prior commitment, a planner would have no motive at those later dates to choose a policy consistent with the anticipations that it was desirable to create at date t_0.

However, the degree of advance commitment that is necessary to bring about an optimal equilibrium is of only a limited sort. Let $x_t \equiv (\Pi_t, Y_t, \Delta_t)$, $X_t \equiv (F_t, K_t)$, and let $\mathcal{F}(\xi_t)$ be the set of values for (Δ_{t-1}, X_t) such that there exist paths $\{x_T\}$ for dates $T \geq t$ that satisfy (1.19) and (1.20) for each T, that are consistent with the specified values for the elements of X_t, and that imply a well-defined value for the objective U_t defined in (1.8).[17] Furthermore, for any $(\Delta_{t-1}, X_t) \in \mathcal{F}(\xi_t)$, let $V(\Delta_{t-1}, X_t; \xi_t)$

[15] At least, this is the case if one restricts attention to those equilibrium in which inflation and output remain forever within certain neighborhoods of the steady-state values defined below. We are here concerned solely with the choice of an optimal policy from among those policies consistent with a nearby equilibrium of this kind, as this is the problem to which our approximation technique may be applied.

[16] Here the definitions (1.14) – (1.15) are understood to have been substituted for F_t and K_t in equation (1.19).

[17] In the notation $\mathcal{F}(\xi_t)$, ξ_t refers to the state of the world at date t, *i.e.*, to a complete specification

denote the maximum attainable value of U_t among the state-contingent paths that satisfy the constraints just mentioned. Then the t_0-optimal plan can be obtained as the solution to the following two-stage optimization problem.

In the first stage, values of the endogenous variables x_{t_0} and state-contingent commitments $X_{t_0+1}(\xi_{t_0+1})$ for the following period, are chosen so as to maximize an objective defined below. Then in the second stage, the equilibrium evolution from period t_0+1 onward is chosen to solve the maximization problem that defines the value function $V(\Delta_{t_0}, X_{t_0+1}; \xi_{t_0+1})$, given the state of the world ξ_{t_0+1} and the precommitted values for X_{t_0+1} associated with that state.

In defining the objective for the first stage of this equivalent formulation of the Ramsey problem, it is useful to let $\Pi(F, K)$ denote the value of Π_t that solves (1.19) for given values of F_t and K_t. We also define the functional relationships

$$\hat{J}[x, X(\cdot)](\xi_t) \equiv U(Y_t, \Delta_t; \xi_t) + \beta E_t V(\Delta_t, X_{t+1}; \xi_{t+1}),$$

$$\hat{F}[x, X(\cdot)](\xi_t) \equiv (1-\tau_t) f(Y_t; \xi_t) + \alpha\beta E_t\{\Pi(F_{t+1}, K_{t+1})^{\theta-1} F_{t+1}\},$$

$$\hat{K}[x, X(\cdot)](\xi_t) \equiv k(Y_t; \xi_t) + \alpha\beta E_t\{\Pi(F_{t+1}, K_{t+1})^{\theta(1+\omega)} K_{t+1}\},$$

where $f(Y;\xi)$ and $k(Y;\xi)$ are defined in (1.16) and (1.17).

Then in the first stage, x_{t_0} and $X_{t_0+1}(\cdot)$ are chosen so as to maximize $\hat{J}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0})$ over values of x_{t_0} and $X_{t_0+1}(\cdot)$ such that

(i) Π_{t_0} and Δ_{t_0} satisfy (1.20);

(ii) the values

$$F_{t_0} = \hat{F}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}), \tag{1.21}$$

$$K_{t_0} = \hat{K}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}) \tag{1.22}$$

satisfy

$$\Pi_{t_0} = \Pi(F_{t_0}, K_{t_0}); \tag{1.23}$$

and

(iii) the choices $(\Delta_{t_0}, X_{t_0+1}) \in \mathcal{F}$ for each possible state of the world ξ_{t_0+1}.

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of all information that is available at that date about both the current exogenous disturbances and the joint probability distribution of all future disturbances. Under the assumption that the state vector ξ_t is Markovian, we can use the same notation ξ_t for a summary of all exogenous disturbances in period t and the state of the world in period t. The argument ξ_t of the value function $V(\Delta_{t-1}, X_t; \xi_t)$ has the same interpretation.

The following result can then be established, as shown in the appendix.

PROPOSITION 1. Given Δ_{t_0-1}, let the process $\{x_t\}$ be determined by (i) choosing x_{t_0} and state-contingent commitments $X_{t_0+1}(\xi_{t_0+1})$ to solve the first-stage problem just stated, and (ii) for each possible state of the world ξ_{t_0+1}, choosing the evolution of x_t for $t \geq t_0+1$ so as to maximize U_{t_0+1}, among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0+1$, given Δ_{t_0}, and that are also consistent with the value of $X_{t_0+1}(\xi_{t_0+1})$ determined in the first stage. Then the process $\{x_t\}$ represents a *Ramsey policy;* that is, it maximizes U_{t_0} among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}.

The optimization problem in stage two of this reformulation of the Ramsey problem is of the same form as the Ramsey problem itself, except that there are additional constraints associated with the precommitted values for the elements of $X_{t_0+1}(\xi_{t_0+1})$. Let us consider a problem like the Ramsey problem just defined, looking forward from some period t_0, except under the constraints that the quantities X_{t_0} must take certain given values, where $(\Delta_{t_0-1}, X_{t_0}) \in \mathcal{F}(t_0)$. This constrained problem can similarly be expressed as a two-stage problem of the same form as above, with an identical stage two problem to the one described above. Stage two of this constrained problem is thus of exactly the same form as the problem itself. Hence the constrained problem has a recursive form, even though the original Ramsey problem did not. This is shown by the following proposition, also proved in the appendix.

PROPOSITION 2. Given some $(\Delta_{t_0-1}, X_{t_0}) \in \mathcal{F}(t_0)$, consider the sequential decision problem in which in each period $t \geq t_0$, $(x_t, X_{t+1}(\cdot))$ are chosen to maximize $\hat{J}[x_t, X_{t+1}(\cdot)](\xi_t)$, subject to constraints (i) – (iii) of the "first stage" problem stated above, given the predetermined state variable Δ_{t-1} and the precommitted values X_t. Then the process $\{x_t\}$ that is chosen in this way is the process that maximizes U_{t_0} among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}, and also consistent with the specified values X_{t_0}.

Our aim here is to characterize policy that solves the constrained optimization problem with which Proposition 2 is concerned *i.e.*, policy that is optimal from some date t onward given precommitted values for X_t. Because of the recursive form of this problem, it is possible for a commitment to a time-invariant policy rule from date t onward to implement an equilibrium that solves the problem, for some specification of the initial commitments X_t. A time-invariant policy rule with this property is said by Woodford (2003, chapter 7) to be "optimal from a timeless perspective."[18] Such a rule is one that a policymaker that solves a traditional Ramsey problem would be willing to commit to *eventually* follow, though the solution to the Ramsey problem involves different behavior initially, as there is no need to internalize the effects of prior anticipation of the policy adopted for period t_0.[19] One might also argue that it is desirable to commit to follow such a rule immediately, even though such a policy would not solve the (unconstrained) Ramsey problem, as a way of demonstrating one's willingness to accept constraints that one wishes the public to believe that one will accept in the future.

2 A Linear-Quadratic Approximate Problem

In fact, we shall here characterize the solution to this problem (and similarly, derive optimal time-invariant policy rules) only for initial conditions near certain steady-state values, allowing us to use local approximations in characterizing optimal policy.[20] We establish that these steady-state values have the property that if one starts from initial conditions close enough to the steady state, and exogenous disturbances thereafter are small enough, the optimal policy subject to the initial commitments remains forever near the steady state. Hence our local characterization describes the *long run* character of Ramsey policy, in the event that disturbances are small

[18]See also Woodford (1999) and Giannoni and Woodford (2002).

[19]In the present model, Ramsey policy involves an initial positive rate of inflation, even in the absence of any shocks, even though in the long run it involves a commitment to maintain a zero inflation rate on average. This is because welfare is increased by exploiting the Phillips curve to increase output through an inflationary policy initially; but it is not optimal to create the anticipation that one will behave in this way later, owing to the adverse effects of the anticipated inflation on earlier periods' inflation/output tradeoffs. See Woodford (2003, chapter 7) for further discussion.

[20]Local approximations of the same sort are often used in the literature in numerical characterizations of Ramsey policy. Strictly speaking, however, such approximations are valid only in the case of initial commitments X_{t_0} near enough to the steady-state values of these variables, and the t_0- optimal (Ramsey) policy need not involve values of X_{t_0} near the steady-state values, even in the absence of random disturbances.

enough.[21] Of greater interest here, it describes policy that is optimal from a timeless perspective in the event of small disturbances.

We first must show the existence of a steady state, *i.e.,* of an optimal policy (under appropriate initial conditions) that involves constant values of all variables. To this end we consider the purely deterministic case, in which the exogenous disturbances $\bar{C}_t$,G_t,$\bar{H}_t$,A_t,μ_t^w,τ_t each take constant values $\bar{C}, \bar{H}, \bar{A}, \bar{\mu}^w, \bar{\tau} > 0$, $\bar{G} \geq 0$ for all $t \geq t_0$. We wish to find an initial degree of price dispersion Δ_{t_0-1} and initial commitments $X_{t_0} = \bar{X}$ such that the solution to the problem defined in Proposition 2 involves a constant policy $x_t = \bar{x}$, $X_{t+1} = \bar{X}$ each period, in which $\bar{\Delta}$ is equal to the initial price dispersion. We show in the appendix that the first-order conditions for this problem admit a steady-state solution of this form, and we verify below that (when our parameters satisfy certain bounds) the second-order conditions for a local optimum are also satisfied.

We show that $\bar{\Pi} = 1$(zero inflation), and correspondingly that $\bar{\Delta} = 1$(zero price dispersion). [22] We may furthermore assume without loss of generality that the constant values of $\bar{C}$ and $\bar{H}$ are chosen so that in the optimal steady state, $C_t = \bar{C}$ and $H_t = \bar{H}$ each period.[23]

We next wish to characterize the optimal responses to small perturbations of the initial conditions and small fluctuations in the disturbance processes around the above values. To do this, we compute a linear-quadratic approximate problem, the solution to which represents a linear approximation to the solution to the policy problem defined in Proposition 2. An important advantage of this approach is that it allows direct comparison of our results with those obtained in other analyses of optimal monetary stabilization policy. Other advantages are that it makes it straightforward to verify whether the second-order conditions hold that are required in order for a solution to our first-order conditions to be at least a local optimum (see section xx), and that it provides us with a welfare measure with which to rank alternative

[21] We show below in section xx that Ramsey policy converges asymptotically to the steady state of the constrained problem, so that the solution to the approximate problem approximates the response to small shocks under the Ramsey policy, at dates long enough after t_0.

[22] Our conclusion that the optimal steady-state inflation rate is zero can be generalized for other price-setting mechanisms and general preference specifications as it is shown in section xx, and for the case in which only distorting taxes are available as in Benigno and Woodford (2003a).

[23] Note that we may assign arbitrary positive values to $\bar{C}, \bar{H}$ without changing the nature of the implied preferences, as long as the value of λ is appropriately adjusted.

sub-optimal policies, in addition to allowing computation of the optimal policy.

We begin by computing a Taylor-series approximation to our welfare measure (1.8), expanding around the steady-state allocation defined above, in which $y_t(i) = \bar{Y}$ for each good at all times and $\xi_t = 0$ at all times.[24] As a second-order (logarithmic) approximation to this measure, we obtain[25]

$$\begin{aligned} U_{t_0} &= \bar{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Phi \hat{Y}_t - \frac{1}{2} u_{yy} \hat{Y}_t^2 + \hat{Y}_t u_{y\xi} \xi_t - u_{\Delta} \hat{\Delta}_t \\ &+ \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{2.1}$$

where $\hat{Y}_t \equiv \log(Y_t/\bar{Y})$ and $\hat{\Delta}_t \equiv \log \Delta_t$ measure deviations of aggregate output and the price dispersion measure from their steady-state levels, the term "t.i.p." collects terms that are independent of policy (constants and functions of exogenous disturbances) and hence irrelevant for ranking alternative policies, and $||\xi||$ is a bound on the amplitude of our perturbations of the steady state.[26] Here the coefficient

$$\Phi \equiv 1 - \frac{\theta - 1}{\theta} \frac{1 - \bar{\tau}}{\bar{\mu}^w} < 1$$

measures the steady-state wedge between the marginal rate of substitution between consumption and leisure and the marginal product of labor, and hence the inefficiency of the steady-state output level $\bar{Y}$. The coefficients $u_{yy}, u_{y\xi}$ and u_{Δ} are defined in the appendix.

In addition, we can take a second-order approximation to equation (1.20) and

[24] Here the elements of ξ_t are assumed to be $\bar{c}_t \equiv \log(\bar{C}_t/\bar{C}), \bar{h}_t \equiv \log(\bar{H}_t/\bar{H}), a_t \equiv \log(A_t/\bar{A}), \hat{\mu}_t^w \equiv \log(\mu_t^w/\bar{\mu}^w), \hat{G}_t \equiv (G_t - \bar{G})/\bar{Y}$, and $\hat{\tau}_t \equiv (\tau_t - \bar{\tau})/\bar{\tau}$, so that a value of zero for this vector corresponds to the steady-state values of all disturbances. The perturbation $\hat{G}_t$ is not defined to be logarithmic so that we do not have to assume positive steady-state value for this variable.

[25] See the appendix for details. Our calculations here follow closely those of Woodford (2003, chapter 6).

[26] Specifically, we use the notation $\mathcal{O}(||\xi||^k)$ as shorthand for $\mathcal{O}(||\xi, \hat{\Delta}_{t_0-1}^{1/2}, \hat{X}_{t_0}||^k)$, where in each case hats refer to log deviations from the steady-state values of the various parameters of the policy problem. We treat $\hat{\Delta}_{t_0}^{1/2}$ as an expansion parameter, rather than $\hat{\Delta}_{t_0}$ because (1.20) implies that deviations of the inflation rate from zero of order ϵ only result in deviations in the dispersion measure Δ_t from one of order ϵ^2. We are thus entitled to treat the fluctuations in Δ_t as being only of second order in our bound on the amplitude of disturbances, since if this is true at some initial date it will remain true thereafter.

integrate it to obtain

$$\sum_{t=t_0}^{\infty}\beta^{t-t_0}\ \hat{\Delta}_t = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)}\theta(1+\omega)(1+\omega\theta)\sum_{t=t_0}^{\infty}\beta^{t-t_0}\ \frac{\pi_t^2}{2}+\text{t.i.p.}+\mathcal{O}(||\xi||^3). \quad (2.2)$$

Substituting (2.2) into (2.1), we can then approximate our welfare measure by

$$\begin{aligned} U_{t_0} &= \bar{Y}\bar{u}_c \cdot E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}[\Phi\hat{Y}_t - \frac{1}{2}u_{yy}\hat{Y}_t^2 + \hat{Y}_t u_{y\xi}\xi_t - \frac{1}{2}u_\pi\pi_t^2] \\ &\quad +\text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \quad (2.3)$$

for a certain coefficient $u_\pi > 0$ defined in the appendix. Note that we can now write our stabilization objective purely in terms of the evolution of the aggregate variables $\{\hat{Y}_t, \pi_t\}$ and the exogenous disturbances.

We note that when $\Phi > 0$, there is a non-zero linear term in (2.3), which means that we cannot expect to evaluate this expression to second order using only an approximate solution for the path of aggregate output that is accurate only to first order. Thus we cannot determine optimal policy, even up to first order, using this approximate objective together with approximations to the structural equations that are accurate only to first order. Rotemberg and Woodford (1997) avoid this problem by assuming an output subsidy (*i.e.,* a value $\bar{\tau} < 0$) of the size needed to ensure that $\Phi = 0$. Here we wish to relax this assumption. We show here that an alternative way of dealing with this problem is to use a second-order approximation to the aggregate-supply relation to eliminate the linear terms in the quadratic welfare measure. We show in the appendix that to second order, equation (1.19) can be written

$$\begin{aligned} V_t &= \kappa(\hat{Y}_t + c_\xi\xi_t + \frac{1}{2}c_{yy}\hat{Y}_t^2 - \hat{Y}_t c_{y\xi}\xi_t + \frac{1}{2}c_\pi\pi_t^2) + \beta E_t V_{t+1} \\ &\quad +\text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3). \end{aligned} \quad (2.4)$$

Here the notation "s.o.t.i.p." indicates terms independent of policy that are entirely of second or higher order, and we have defined

$$V_t \equiv \pi_t + \frac{1}{2}v_\pi\pi_t^2 + v_z\pi_t Z_t,$$

where

$$Z_t \equiv E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}[z_y\hat{Y}_T + z_\pi\pi_T + z_\xi\xi_T];$$

for certain coefficients defined in the appendix. Note that to first order (2.4) reduces simply to

$$\pi_t = \kappa[\hat{Y}_t + c_\xi \xi_t] + \beta E_t \pi_{t+1}, \tag{2.5}$$

for a certain coefficient $\kappa > 0$. This is the familiar "New Keynesian Phillips curve" relation.

Integrating forward equation (2.4), we obtain a relation of the form

$$V_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \kappa[\hat{Y}_t + \frac{1}{2} c_{yy} \hat{Y}_t^2 - \hat{Y}_t c_{y\xi} \xi_t + \frac{1}{2} c_\pi \pi_t^2] + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \tag{2.6}$$

We can then use (2.6) to write the discounted sum of output terms in (2.3) as a function of purely quadratic terms, up to a residual of third order. As shown in the appendix, we can rewrite (2.3) as

$$U_{t_0} = -\Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2} \pi_t^2 + \frac{q_y}{2} (\hat{Y}_t - \hat{Y}_t^*)^2 \right\} + T_{t_0} + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \tag{2.7}$$

where [27]

$$\Omega \equiv \bar{Y} u_c > 0,$$

$$q_\pi \equiv \frac{\theta}{\kappa}[(\omega + \sigma^{-1}) + \Phi(1 - \sigma^{-1})], \tag{2.8}$$

$$q_y \equiv \omega + \sigma^{-1} + \Phi(1 - \sigma^{-1}) - \frac{\Phi \sigma^{-1}(s_C^{-1} - 1)}{\omega + \sigma^{-1}}, \tag{2.9}$$

$$\hat{Y}_t^* = \omega_1 \hat{Y}_t^n - \omega_2 \hat{G}_t + \omega_3 \hat{u}_t^w + \omega_4 \hat{\tau}_t, \tag{2.10}$$

and

$$\hat{Y}_t^n \equiv -c_\xi \xi_t = \frac{\sigma^{-1} g_t + \omega q_t - \hat{\mu}_t^w - \omega_\tau \hat{\tau}_t}{(\omega + \sigma^{-1})},$$

in which expressions

$$\omega_1 = q_y^{-1}[(\omega + \sigma^{-1}) + \Phi(1 - \sigma^{-1})],$$

$$\omega_2 = \frac{\Phi s_C^{-1} \sigma^{-1}}{(\omega + \sigma^{-1})^2 + \Phi[(1 - \sigma^{-1})(\omega + \sigma^{-1}) - (s_C^{-1} - 1)\sigma^{-1}]},$$

$$\omega_3 \equiv \frac{1 - \Phi}{(\omega + \sigma^{-1}) + \Phi[(1 - \sigma^{-1}) - (s_C^{-1} - 1)\sigma^{-1}(\omega + \sigma^{-1})^{-1}]},$$

[27] In what follows, the following definitions have been used: $\sigma^{-1} \equiv \tilde{\sigma}^{-1} s_C^{-1}$ with $s_C \equiv \bar{C}/\bar{Y}$; $\omega q_t \equiv \nu \bar{h}_t + \phi(1+\nu) a_t$; $g_t \equiv \hat{G}_t + s_C \bar{c}_t$; $\omega_\tau \equiv \bar{\tau}/(1-\bar{\tau})$; $\kappa \equiv (1-\alpha\beta)(1-\alpha)(\omega+\sigma^{-1})/[\alpha(1+\theta\omega)]$.

$$\omega_4 \equiv \frac{\omega_\tau}{(\omega + \sigma^{-1}) + \Phi[(1-\sigma^{-1}) - (s_C^{-1} - 1)\sigma^{-1}(\omega + \sigma^{-1})^{-1}]}.$$

Here $\hat{Y}_t^n$ represents a log-linear approximation to the "natural rate of output," *i.e.*, the flexible-price equilibrium level of output (Woodford, 2003, chap. 3); in terms of this notation, the log-linear aggregate supply relation (2.5) can be written as

$$\pi_t = \kappa[\hat{Y}_t - \hat{Y}_t^n] + \beta E_t \pi_{t+1}. \tag{2.11}$$

The term $T_{t_0} \equiv \Phi \bar{Y} \bar{u}_c \kappa^{-1} V_{t_0}$ is a transitory component defined in the appendix.

Once again, we are interested in characterizing optimal policy from a timeless perspective. We observe from the form of the structural relations (2.4) and the definition of V_t that the aspects of the expected future evolution of the endogenous variables that affect the feasible set of values for inflation, output in any period t can be summarized (in our second-order approximation to the structural relations) by the expected values of V_{t+1}, Z_{t+1}. Hence the only commitments regarding future outcomes that can be of value in improving stabilization outcomes in period t can be summarized by commitments at t regarding the state-contingent values of those two variables in the following period. It follows that we are interested in characterizing optimal policy from any date t_0 onward subject to the constraint that given values for V_{t_0}, Z_{t_0} be satisfied,[28] in addition to the constraints represented by the structural equations.

But given predetermined values for V_{t_0} the value of the transitory component T_{t_0} is predetermined. Hence, over the set of admissible policies, higher values of (2.7) correspond to lower values of

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2} \pi_t^2 + \frac{q_y}{2} (\hat{Y}_t - \hat{Y}_t^*)^2 \right\}. \tag{2.12}$$

It follows that we may rank policies in terms of the implied value of the discounted quadratic loss function (2.12). Because this loss function is purely quadratic (*i.e.*, lacking linear terms), it is possible to evaluate it to second order using only a first-order approximation to the equilibrium evolution of inflation and output under a given policy. Hence the log-linear approximate structural relation (2.5) (or equivalently,

[28]Note that a specification of initial values for these two variables corresponds, in our quadratic approximation to the structural equations, to a specification of initial values for the three variables F_{t_0}, K_{t_0} in section 1.

(2.11)) is sufficiently accurate for our purposes. Similarly, it suffices that we use log-linear approximations to the variable V_{t_0} in describing the initial commitments, which are given by $\hat{V}_{t_0} = \pi_{t_0}$. Then an optimal policy from a timeless perspective is a policy from date t_0 onward that minimizes the quadratic loss function (2.12) subject to the constraints implied by the linear structural relation (2.11) holding in each period $t \geq t_0$ and subject also to the constraints that a certain predetermined value for $\hat{V}_{t_0}$ be achieved.[29] This last constraint may equivalently be expressed as a constraint on the initial inflation rate,

$$\pi_{t_0} = \bar{\pi}_{t_0}. \tag{2.13}$$

(The definition of the constraint value $\bar{\pi}_{t_0}$ under a policy that is optimal from a timeless perspective is discussed further in section xx below.)

The policy objective (2.12) now depends only on the evolution of the inflation rate and the welfare-relevant output gap

$$x_t \equiv \hat{Y}_t - \hat{Y}_t^*.$$

It is useful to write the linear constraints implied by our model's structural equations in terms of the welfare-relevant output gap as well. The aggregate-supply relation (2.11) can be alternatively expressed as

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t, \tag{2.14}$$

where u_t is a composite "cost-push" term, indicating the degree to which the exogenous disturbances preclude simultaneous stabilization of inflation and the welfare-relevant output gap. In terms of our previous notation for the exogenous disturbances in the model, this is given by

$$\begin{aligned} u_t &\equiv \kappa(\hat{Y}_t^* - \hat{Y}_t^n) \\ &= \kappa(\omega_1 - 1)\hat{Y}_t^n - \kappa\omega_2 \hat{G}_t + \kappa\omega_3 \hat{u}_t^w + \kappa\omega_4 \hat{\tau}_t. \end{aligned}$$

It is important for the discussion below to note that pure markup shocks are not the only source of movements in the cost-push term u_t.

[29]The constraint associated with a predetermined value for Z_{t_0} can be neglected, in a first-order characterization of optimal policy, because the variable Z_t does not appear in the first-order approximation to the aggregate-supply relation.

We have thus shown that an objective for policy of the form (xx), as discussed in the introduction, can indeed be justified on welfare-theoretic grounds. This requires that the "output gap" in such an objective be interpreted in the way defined here, *i.e.*, as the percentage deviation of output from a variable target level of output that depends on the evolution of exogenous disturbances of many sorts. (There is thus no reason, in general, for the welfare-theoretic target level of output to correspond to a smooth trend.) We have also seen that exogenous disturbances may indeed preclude simultaneous stabilization of inflation and the welfare-relevant output gap; the extent to which this is true depends on the degree of variability of the disturbance term u_t defined above. We now turn to the consequences of this characterization for the nature of optimal policy.

3 Optimal Inflation Stabilization

We now use our linear-quadratic approximate policy problem to characterize optimal policy in the event of small enough disturbances. We begin by establishing conditions under which the second-order conditions for loss minimization are satisfied, so that the first-order conditions determine a loss-minimizing policy, and hence approximate at least a local welfare maximum. These are also conditions under which welfare cannot be increased (at least locally) by arbitrary randomization of policy. We then use the first-order conditions to characterize the optimal responses of inflation and output to exogenous disturbances, and discuss the conditions under which optimal policy corresponds to complete price stability.

3.1 Undesirability of Policy Randomization

We have shown in the previous section that our approximate policy problem consists of choosing processes $\{\pi_t, \hat{Y}_t\}$ for dates $t \geq t_0$ to minimize the loss function (2.12), subject to the constraint that the log-linear approximate aggregate supply relation (2.14) hold each period, and that the initial inflation rate satisfy a constraint of the form (2.13). We first consider whether a solution to the first-order conditions associated with this problem necessarily represents a loss minimum. This is necessarily true if the loss function is convex, as it will be if $q_\pi, q_y > 0$; but as we shall see, our approximate loss function is not necessarily convex, yet our LQ approximation may nonetheless suffice to characterize (locally) optimal policy. Here we examine the somewhat weaker conditions under which this will still be true.

As a closely related question, we consider the issue of whether purely random policy — randomization of policy by the monetary authority, uncorrelated with any random variation in economic "fundamentals" — can be welfare-improving. Again, in the case of a convex loss function, of the kind conventionally assumed in analyses of monetary stabilization policy with *ad hoc* objectives, it can be shown that arbitrary randomization is never optimal. But if our approximate loss function need not be convex, the answer is not obvious, and Dupor (2003) exhibits a general-equilibrium model with sticky prices in which randomization of monetary policy can be welfare-improving. Here we use our LQ approximation method to establish general conditions under which a result like Dupor's will obtain in a model with Calvo-style staggered pricing.

We begin with the simpler question of the desirability of policy randomization. Suppose that we begin with some possible equilibrium processes $\{\pi_t, \hat{Y}_t\}$ consistent with constraint (2.13), and then consider the effect of perturbing this equilibrium by adding terms that depend on a sunspot realization v_t at some date $t > t_0$. The variable v_t is assumed to have conditional expectation zero at date t_0 and variance $\sigma_v^2 > 0$, and to be distributed independently of the processes $\{\pi_t, \hat{Y}_t\}$ and of fundamental processes such as $\{\hat{Y}_t^*, \hat{Y}_t^n\}$. Suppose that the occurrence of this shock adds a contribution $\psi_j v_t$ to π_{t+j}, for each $j \geq 0$, where $\{\psi_j\}$ is an arbitrary bounded sequence of coefficients. (This is the effect of the sunspot in the linear approximation to the solution under a given random policy which suffices for our computation of a second-order approximation to welfare.) It then follows from (2.14) that the shock also adds a contribution $\kappa^{-1}(\psi_j - \beta\psi_{j+1})v_t$ to $\hat{Y}_{t+j}$ for each $j \geq 0$.

The response to this sunspot thus adds terms to the loss measure (2.12) equal to

$$\beta^t \sigma_v^2 \sum_{j=0}^{\infty} \beta^j \left[q_\pi \psi_j^2 + q_y \left(\frac{\psi_j - \beta\psi_{j+1}}{\kappa} \right)^2 \right]. \tag{3.1}$$

Randomization of monetary policy is (locally) undesirable if and only if this expression is positive in the case of all possible non-zero bounded sequences $\{\psi_j\}$. This is turn holds if and only if the quadratic form

$$||\psi|| \equiv \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Lambda_{ij} \psi_i \psi_j \tag{3.2}$$

is *positive definite*for appropriately defined coefficients Λ_{ij}. (See the appendix for details.)

If (3.2) is positive definite, it similarly follows that no bounded added terms that depend linearly on sunspot realizations in any of the periods $t > t_0$ can lower the expected loss. Hence a loss-minimizing policy must be one that involves no arbitrary randomization. One can also show that the same condition describes the complete set of second-order conditions for loss minimization. In the appendix we prove the following proposition.

PROPOSITION 3. Randomization of monetary policy reduces the expected losses (2.12) — and hence is locally welfare-reducing in the exact problem as well — if and only if the quadratic form (3.2) is positive definite. Furthermore, if and only if this is true, processes $\{\pi_t, \hat{Y}_t\}$ that satisfy the first-order conditions for the LQ optimization problem [discussed further below] represent a loss minimum, and hence an approximation to (at least a local) welfare maximum in the exact problem.

Furthermore, the necessary and sufficient conditions for (3.2) to be positive definite reduce to the following: q_π and q_y are not *both* equal to zero; and *either* (i) $q_y \geq 0$ and

$$q_\pi + (1 - \beta^{1/2})^2 \kappa^{-2} q_y \geq 0, \tag{3.3}$$

holds, or (ii) $q_y \leq 0$ and

$$q_\pi + (1 + \beta^{1/2})^2 \kappa^{-2} q_y \geq 0, \tag{3.4}$$

holds.

Note that in the case that both $q_y, q_\pi \geq 0$, (3.3) is satisfied as long as least one coefficient is strictly positive; thus the case of a convex loss function is one in which the second-order conditions are necessarily satisfied and randomization of policy is necessarily welfare-reducing. However, Proposition 3 shows that the requirement of convexity of the loss function can be weakened while retaining these results.

In fact, in the case of isoelastic functional forms, convexity is likely to obtain for quantitatively reasonable parameter values, even if it is not a necessary consequence of the general assumptions made above. In the isoelastic case, q_y and q_π are given by (2.9) and (2.8) respectively. It follows from this expression and our general assumptions that $q_\pi > 0$, though it remains possible in the isoelastic case for q_y to be negative. Furthermore, one observes that a necessary condition for q_y to be negative is that $s_C < 1/2$, or alternatively that $s_G > 1/2$, which is larger share of government purchases in total demand than is typical of industrial economies.

Even if $q_y < 0$, Proposition 3 shows that randomization of policy will still be welfare-reducing, as long as

$$q_y \geq -\frac{\kappa^2 q_\pi}{(1+\beta^{1/2})^2}. \tag{3.5}$$

Violation of this bound requires an even more extreme role of the government in the economy, though it remains a technical possibility, consistent with our general neoclassical assumptions.[30] We show in section xx, however, that it is possible for randomization to be welfare-improving without such an extremely large share of government purchases in total demand, in the case of more general functional forms.

3.2 The Case for Price Stability

Under certain circumstances, our characterization of the approximate loss function yields immediate conclusions regarding the nature of optimal policy. These are the conditions under which optimal policy involves complete stabilization of the inflation rate at zero, *i.e.*, complete price stability. While the conditions under which this is exactly true are fairly special, they are nonetheless of interest, insofar as price stability may be a good approximation to optimal policy as long as the conditions are not too grossly violated.

The quadratic loss function (2.12) is clearly minimized by a policy under which inflation is zero at all times if two conditions are met: (i) the coefficients of the loss function satisfy $q_y, q_\pi > 0$; and (ii) the exogenous terms $\hat{Y}_t^n$ and $\hat{Y}_t^*$ coincide at all times. Condition (ii) implies that a policy under which inflation is zero at all times will also involve $\hat{Y}_t = \hat{Y}_t^*$ at all times, as a consequence of (2.14).[31] Condition (i)

[30] For given values $0 < \beta < 1$, $\omega \geq 0, \sigma^{-1} > 0$, $\Phi > 0$, $\kappa > 0$, and $\theta > 1$, choice of a value of s_G close enough to 1 — and hence a value of s_C close enough to zero — will make q_y an arbitrarily large negative quantity, while q_π and the other expressions on the right-hand side of (3.5) remain finite. Hence it is possible to find parameter values for which (3.5) is violated.

[31] Here we assume that a policy under which inflation is zero at all times is *feasible*. In the model proposed here, this is necessarily the case as long as disturbances are small enough, so that the nominal interest rate required for an equilibrium with zero inflation is non-negative at all times.

then implies that such an equilibrium necessarily achieves the lowest possible value for expected losses, since expected losses are zero and the loss function is necessarily non-negative.

In fact, condition (i) can be weakened; it suffices that q_y and q_π satisfy the conditions stated in Proposition 3. In the appendix we establish the following result.

PROPOSITION 4. Suppose that $\hat{Y}_t^n = \hat{Y}_t^*$ at all times, and that the conditions stated in Proposition 3 are satisfied. Then the policy that uniquely minimizes (2.12) is the one under which $\pi_t = 0$ at all times, regardless of the realizations of the exogenous disturbances [as long as these are small enough to make such an equilibrium possible].

This means that in the exact model as well, a policy under which inflation is zero at all times is optimal from a timeless perspective. That is, under the initial constraint that $\pi_{t_0} = 0$, expected utility is maximized by a policy under which $\pi_t = 0$ for all $t \geq t_0$.

The condition that $\hat{Y}_t^n = \hat{Y}_t^*$ at all times, assumed in Proposition 4, is not quite so special a situation as might be imagined. It is consistent with the existence of a number of distinct types of independent disturbances, as long as certain model parameters take special values. Comparing the definitions of $\hat{Y}_t^n$ and $\hat{Y}_t^*$ above, one sees that [for the isoelastic case considered in section 2] both expressions will be affected to exactly the same extent by technology shocks, by shocks to household impatience to consume, and by shocks to the disutility of labor supply, in the case that $\omega_1 = 1$. This condition in turn holds if $\Phi(s_C^{-1} - 1) = 0$, which holds if *either* $\Phi = 0$ or $s_G = 0$. Furthermore, both expressions are affected to exactly the same extent by variations in government purchases as well, if in addition $\omega_2 = 0$, which holds if $\Phi = 0$. However, variations in the wage markup or in the level of distorting taxes necessarily affect the two expressions differently, except in a special case that would imply that they are no longer affected in the same way by any disturbances to tastes or technology. We thus obtain the following result.

PROPOSITION 5. Consider a model with the isoelastic functional forms (1.3) – (1.4), and parameter values $\omega \geq 0, \sigma^{-1} > 0$, and suppose that there are random fluctuations in the composite disturbance term $\omega q_t + \sigma^{-1}\bar{c}_t$. [This is generally true if either preferences or technology are random.] Then $\hat{Y}_t^n = \hat{Y}_t^*$ at all times — so that the "cost-push" term in the aggregate-supply relation (2.14) is zero at all times — if and only if (i) there are no random variations in the wage markup or the tax rate ($\hat{\mu}_t^w = \hat{\tau}_t = 0$ at all times); and (ii) *either* (a) the steady-state level of output is efficient ($\Phi = 0$) or (b) there are no government purchases ($G_t = 0$ at all times).

The result that there is no "cost-push" term in the aggregate-supply relation in the case that $\Phi = 0$, as long as there are no markup fluctuations or variations in the level of distorting taxes, has already been obtained in Woodford (2003, chap. 6), following Rotemberg and Woodford (1997). Here there is also a simple intuition for the fact that price stability is optimal, first stated by Goodfriend and King (1997): the model is one in which, if prices were perfectly flexible, the equilibrium allocation of resources would be optimal. Even with staggered price adjustment, a policy that achieves zero inflation at all times leads to an equilibrium allocation of resources that is the same as if prices were flexible; hence the policy is optimal.

More interesting is the conclusion that even when the steady-state is inefficient ($\Phi > 0$), a policy of complete price stability is *still* optimal (from a timeless perspective[32]) in the isoelastic case, as long as there are no government purchases. (The absence of government purchases is actually necessary in order for this case to be isoelastic in the relevant sense; for it is only if $G_t = 0$ that (1.3) implies that the marginal utility of income will be an isoelastic function of the level of output Y_t, and not simply of the level of consumption C_t.)

This result provides an analytical explanation of certain numerical results obtained by Khan *et al.* (2003) in a closely related model.[33] Khan *et al.* assume isoelastic functional forms, as we have, and also calibrate their model so that in the steady state there are no government purchases ($s_G = 0$), even though they consider the effects of small departures of G_t from the steady-state value of zero. When they consider the optimal policy response to a technology shock, and use a linearization method[34]

[32]We discuss the way in which Ramsey policy differs from the timelessly optimal policy in section xx.

[33]The model considered by Khan *et al.*, in the variant that abstracts from monetary frictions, is essentially the same as ours, except for a different form of staggering of pricing decisions: in their model, each price remains in force for exactly two periods (with half of the prices being reoptimized each period), rather than for a random number of periods as in the Calvo model. We discuss the consequences of this alternative form of staggering in section xx below.

[34]The method that they use to compute a linear approximation to optimal policy involves first

to compute a linear approximation to the optimal response — i.e., to compute the derivative of the optimal paths with respect to the amplitude of the technology shock, evaluated at the case of a zero disturbance (the steady state) — they are in effect computing a linear approximation to optimal policy in a model in which there are no government purchases, since they compute a perturbation which does not change the level of government purchases to a steady state with no government purchases. In fact, Khan *et al.* find that the optimal response to a technology shock involves no change in the inflation rate (which continues to equal zero, the optimal steady-state inflation rate in their model as in ours), and a response of output that is the same as would occur in a model with flexible prices (i.e., $\hat{Y}_t = \hat{Y}_t^n$).[35] This is just what Propositions 4 and 5 would imply for our model.

Instead, they find that the optimal response to a variation in government purchases involves some change in the inflation rate, and an output response that differs slightly from the flexible-price equilibrium response. This too is what our analysis would predict, in the case that $\Phi > 0$. Thus our results provide analytical insight into the reason for the numerical results obtained by Khan *et al.* for a particular numerical calibration, which allows us to better understand their degree of generality. On the one hand, we find that their conclusion with regard to technology shocks does not depend on their precise parameter values, except the choice to assume that $s_G = 0$. However, our analysis also indicates that they would not have obtained the same result under a more realistic calibration in which $s_G > 0$; so this simplification was not innocuous. Our further analysis in section xx below also shows that their result would not obtain, in general, in the case of non-isoelastic functional forms, even under the assumption that $s_G = 0$.

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writing the exact (nonlinear) first-order conditions that characterize optimal policy, then linearizing these first-order conditions, and solving the linearized equations. This method yields an identical linear approximation to optimal policy as the solution to our LQ problem though, as we have explained in section 2, we believe there are advantages to proceeding from an LQ approximate policy problem.

[35] King and Wolman (1999) obtain a similar conclusion in a model where government purchases are not considered at all.

3.3 Optimal Responses to "Cost-Push" Disturbances

While in the previous section we have described cases in which complete price stability is optimal, we have also found that this is exactly true only in fairly special cases, when we allow (realistically) for a distorted steady state. In general, the "cost-push" term u_t will be non-zero. This is obviously true if there is time variation in the size of tax distortions or in wage markups, since disturbances of this kind affect the flexible-price equilibrium level of output while they are irrelevant for the efficient allocation of resources. But our results above show that even if there are no disturbances of those types, shocks to tastes or technology, or variations in government purchases, *also* generally give rise to fluctuations in the cost-push term. In any such case, it is not possible simultaneously to fully stabilize both inflation and the welfare-relevant output gap; the optimal trade-off between the two stabilization objectives generally involves some degree of variation in both variables in response to disturbances.

In order to consider optimal policy in this more general case, it suffices that we specify the stochastic process for fluctuations in the composite cost-push term $\{u_t\}$; the underlying source of those fluctuations does not matter, at least as far as the optimal fluctuations in inflation and in the welfare-relevant output gap are concerned. (The optimal responses of other variables, such as output, employment, or private consumption, will instead generally depend on what kind of real disturbances have occurred.) It follows from the approximation introduced in section 2 that a log-linear approximation to the optimal evolution of inflation and the output gap are given by the processes $\{\pi_t, x_t\}$ that minimize (2.12), subject to the constraints that the aggregate-supply relation (2.14) be satisfied each period, and that the initial inflation rate satisfy a constraint of the form (2.13). The solution to this problem plainly depends only on the stochastic evolution of the composite cost-push term. Thus from this point we make treat the specification of the transitory fluctuations $\{u_t\}$ as a primitive.

The form of the optimization problem just stated is the same as in a model where the steady state is assumed to be efficient ($\Phi = 0$); the only differences made by allowing Φ to be positive have to do with the expressions that we have derived for q_π and q_y as functions of underlying model parameters, the expression for u_t as a function of underlying disturbances, and the definition of the welfare-relevant output gap x_t. The solution to the problem is therefore the same (in the case of a given

$\{u_t\}$ process and given values of q_π and q_y) as in the $\Phi = 0$ case treated in Woodford (2003, chap. 7).[36] We recall here some of the main results presented there, which directly apply to the present case as well.

The first-order conditions for the optimization problem just stated are of the form

$$q_\pi \pi_t + \varphi_t - \varphi_{t-1} = 0, \tag{3.6}$$

$$q_y x_t - \kappa \varphi_t = 0, \tag{3.7}$$

for each $t \geq t_0$, where φ_t is the Lagrange multiplier associated with the constraint (2.14) in period t. Bounded processes $\{\pi_t, x_t, \varphi_t\}$ that satisfy (2.14) and (3.6) – (3.7) for each $t \geq t_0$ and are consistent with the initial condition (2.13) represent an optimum. Using (3.6) to eliminate π_t and (3.7) to eliminate x_t,[37] (2.14) becomes an equation for the evolution of the multiplier

$$\beta q_y E_t \varphi_{t+1} - [(1+\beta) q_y + \kappa^2 q_\pi] \varphi_t + q_y \varphi_{t-1} = q_\pi q_y u_t. \tag{3.8}$$

The initial condition (2.13) can similarly be expressed as a constraint on the path of the multipliers

$$\varphi_{t_0} - \varphi_{t_0 - 1} = -q_\pi \bar{\pi}_{t_0}. \tag{3.9}$$

An optimum can then be described by a bounded process $\{\varphi_t\}$ for all dates $t \geq t_0 - 1$ that satisfies (3.8) for each $t \geq t_0$ and is also consistent with (3.9).

Equation (3.8) has a unique bounded solution consistent with (3.9) if and only if the characteristic equation

$$\beta q_y \mu^2 - \left[(1+\beta) q_y + \kappa^2 q_\pi\right] \mu + q_y = 0 \tag{3.10}$$

has exactly one root such that $|\mu| < 1$. This requires that the characteristic equation have real roots, exactly one of which lies in the interval between -1 and 1; this in turn is true if and only if[38] $q_\pi \neq 0$ and

$$\frac{q_y}{q_\pi} > -\frac{\kappa^2}{2(1+\beta)}. \tag{3.11}$$

[36] See also Clarida, Gali and Gertler (1999) for analysis of an LQ problem of this form.

[37] Here we assume that both $q_\pi, q_y \neq 0$. Note that if either q_π or q_y happens to equal zero, optimal policy is easily characterized: it consists simply of the complete stabilization of the variable with the non-zero weight in the loss function.

[38] Note that while we have assumed $q_y \neq 0$ in the above derivation, (3.8), (3.9) and (3.10) are also correct even when $q_y = 0$.

Note that in the case that $\Phi = 0$ (treated in Woodford, 2003, chap. 7), this condition is necessarily satisfied, since in that case $q_\pi, q_y > 0$. We then obtain the following result (details are given in the appendix).

Proposition 6. Suppose that $q_\pi \neq 0$, and that (3.11) is satisfied in addition to the conditions listed in Proposition 3. Then in the case of any small enough value of $\bar{\pi}_{t_0}$, and any sufficiently tightly bounded fluctuations in the cost-push disturbance process $\{u_t\}$, the solution to the optimization problem stated in Proposition 2 involves fluctuations $\{\pi_t, x_t\}$ that remain forever within any given neighborhood of the steady-state values $(0, 0)$. These optimal dynamics are furthermore approximated (arbitrarily well, in the case of tight enough bounds on $\bar{\pi}_{t_0}$ and on the amplitude of the cost-push terms) by the log-linear dynamics corresponding to the unique bounded solution to equations (2.14), (3.6) and (3.7) consistent with initial condition (2.13).

This solution is obtained by solving (3.6) and (3.7) for π_t and x_t respectively, where the multiplier process $\{\varphi_t\}$ is specified recursively by the relation

$$\varphi_t = \mu \varphi_{t-1} - q_\pi \sum_{j=0}^{\infty} \beta^j \mu^{j+1} E_t u_{t+j}. \tag{3.12}$$

Here μ is the root of (3.10) that satisfies $-1 < \mu < 1$, and the initial value φ_{t_0-1} is chosen so that that the solution is consistent with (2.13).

In the isoelastic case, as discussed above, $q_\pi > 0$. One can then show furthermore that condition (3.11) implies condition (3.5), though the former condition is stronger.[39] Hence it suffices that (3.11) hold in order for Proposition 6 to apply. Since this is necessarily satisfied if $q_y \geq 0$, it also follows from our discussion above that if $s_G \leq 1/2$, the condition is necessarily satisfied. Thus in the isoelastic case, Proposition 6 necessarily applies, unless government purchases are a large share of

[39] Whenever (3.5) is satisfied, so that a bounded solution to the first-order conditions would correspond to an optimum, there is necessarily no more than one bounded solution. However, there might be *no* bounded solution, as the optimal policy might involve mildly explosive dynamics. This is the case in which (3.5) is satisfied though (3.11) is not. We do not wish to consider such cases here, as our local LQ approximation to the policy problem could not be guaranteed to remain an accurate approximation in such a case. Hence we shall require that the stronger condition (3.11) be satisfied. In the case of an *exact* LQ problem, this condition would not be required in order for (3.8) to determine a well-defined optimal policy.

total output. (But once again, it remains possible for the condition not to hold; indeed, it is possible for (3.11) to fail even though (3.5) is satisfied.)

As an example of the implications of Proposition 6, consider the case of exogenous fluctuations in the level of government purchases, according to a first-order autoregressive process of the form

$$\hat{G}_t = \rho_G \hat{G}_{t-1} + \epsilon_t^G, \tag{3.13}$$

where $0 \leq \rho_G < 1$ and $\{\epsilon_t^G\}$ is an i.i.d., bounded, mean-zero exogenous shock process. It follows from the definition of the cost-push term in section xx above that in this case, $u_t = \gamma_G \hat{G}_t$, with a coefficient

$$\gamma_G \equiv -\kappa \Phi \frac{\sigma^{-1}}{(\omega + \sigma^{-1}) q_y}.$$

In this case, (3.12) reduces to

$$\varphi_t = \mu \varphi_{t-1} + \phi_G \hat{G}_t,$$

where

$$\phi_G \equiv -\frac{q_\pi \mu \gamma_G}{1 - \beta \mu \rho_G}.$$

It then follows that an innovation ϵ_t^G to the level of government purchases affects the current level and expected future path of the Lagrange multiplier by an amount

$$E_t \varphi_{t+j} - E_{t-1} \varphi_{t+j} = \frac{\mu^{j+1} - \rho_G^{j+1}}{\mu - \rho_G} \phi_G \epsilon_t^G$$

for each $j \geq 0$. Given this impulse response for the multiplier, (3.6) – (3.7) can be used to derive corresponding impulse responses for prices and the output gap,

$$E_t p_{t+j} - E_{t-1} p_{t+j} = -\frac{1}{q_\pi} \frac{\mu^{j+1} - \rho_G^{j+1}}{\mu - \rho_G} \phi_G \epsilon_t^G, \tag{3.14}$$

$$E_t x_{t+j} - E_{t-1} x_{t+j} = \frac{\kappa}{q_y} \frac{\mu^{j+1} - \rho_G^{j+1}}{\mu - \rho_G} \phi_G \epsilon_t^G, \tag{3.15}$$

where in (3.14) we use the notation $p_t \equiv \log P_t$.

If we further specialize to the case in which $\bar{G} = 0$, so that $s_C = 1$ (as in the calibration of Khan *et al.*, 2003), then in the case of any $\Phi > 0$ we have

$$q_y = \omega + \Phi + \sigma^{-1}(1 - \Phi) > 0,$$

$$q_\pi = \frac{\theta}{\kappa} q_y > 0,$$

as a consequence of which one can show that $0 < \mu < 1$. We also observe in this case that $\gamma_G < 0$, as a result of which $\phi_G > 0$. It then follows that each of the coefficients of the impulse response function (3.14) is negative, while each of the coefficients of the impulse response function (3.15) is positive. That is, an unexpected increase in government purchases results in a decrease in prices and an increase in the (welfare-relevant) output gap; both impulse responses return asymptotically to zero, without ever overshooting their long-run levels.

This provides us with an analytical explanation of the results of Khan *et al.* (2003) in a closely related model. They also find that the optimal response to an increase in government purchases involves a temporary reduction in prices, together with a greater contraction of private consumption (and a smaller increase in output) than would occur in the flexible-price equilibrium, or than would result from a monetary policy that completely stabilized inflation. Our analytical results here yield the same conclusion. Because $\gamma_G < 0$, an increase in government purchases causes a negative "cost-push shock," meaning that it is not possible to maintain $\hat{Y}_t$ equal to $\hat{Y}_t^*$ without deflation (as $\hat{Y}_t^*$ rises less than does the natural rate $\hat{Y}_t^n$). The optimal tradeoff between the objectives of inflation stabilization and output-gap stabilization requires one to accept some deflation, though not as much as would be required to maintain $\hat{Y}_t$ equal to $\hat{Y}_t^*$.

This involves an increase in the welfare-relevant output gap, and since $\hat{Y}_t^* = \psi_G \hat{G}_t$, where

$$\psi_G = \frac{\sigma^{-1}}{\omega + \sigma^{-1}} \frac{\omega + \sigma^{-1}(1-\Phi)}{\omega + \Phi + \sigma^{-1}(1-\Phi)} > 0,$$

the target level of output also increases; hence output increases relative to trend in response to such a shock. Nonetheless, optimal policy involves output temporarily *lower* than the flexible-price equilibrium level $\hat{Y}_t^n$, as found by Khan *et al.* The price-level response (3.14) implies that $E_t p_{t+1}$ falls by an amount that is $\mu + \rho_G < 2$ times as large as the decline in p_t; hence $E_t \pi_{t+1}$ does not decline by as much as does π_t (if it falls at all). It then follows from (2.11) that $\hat{Y}_t - \hat{Y}_t^n$ must fall in response to a positive innovation ϵ_t^G. Thus output rises less (at least in the period of the shock) under optimal policy than it would in a flexible-price equilibrium; or alternatively, consumption falls by more than it would in a flexible-price equilibrium, as reported by Khan *et al.* Our results for a model with Calvo pricing are thus qualitatively similar to theirs for a model with two-period price commitments, and we are also able to obtain precise analytical expressions for the size of the effects in question.

4 The General Case

In this section we wish to relax some of the assumptions of the previous sections. We continue to assume that each household seeks to maximize the level of expected utility as in (1.1), but now, as in the model of Kimball (1995), the consumption aggregate C_t is implicitly defined by a relation of the form

$$\int_0^1 \psi(c_t(i)/C_t)di = 1, \tag{4.1}$$

where $\psi(x)$ is an increasing, strictly concave function satisfying $\psi(1) = 1$. Under the previous assumption of Dixit-Stiglitz preferences, we had $\psi(x) = x^{(\theta-1)/\theta}$. The demand curve for good i is then implicitly defined by

$$\psi'\left(\frac{y_t(i)}{Y_t}\right) = \psi'(1)\frac{p_t(i)}{P_t}$$

which allows us to define the function $z(x) = {\psi'}^{-1}(\psi'(1)x)$. The price index P_t is implicitly defined by

$$\int_0^1 h(p_t(i)/P_t)di = 1, \tag{4.2}$$

where $h(p) \equiv p{\psi'}^{-1}(\psi'(1)p)$ is a positive-valued, decreasing function.

We also now allow for preferences of the more general form

$$\tilde{u}(C_t;\xi_t) = \tilde{u}(C_t;\xi_{c,t}), \tag{4.3}$$

$$\tilde{v}(H_t;\xi_t) = \tilde{v}(H_t;\xi_{h,t}), \tag{4.4}$$

where $\{\xi_{c,t},\xi_{h,t}\}$ are bounded exogenous scalar disturbance process, and for a production technology of the more general form

$$y_t(i) = A_t f(h_t(i))$$

for each of the differentiated goods. We assume only that our utility and production functions are three times differentiable, and that they satisfy the standard neoclassical assumptions regarding monotonicity and concavity.

In the general case, it is useful to introduce the notation $\tilde{\sigma}^{-1} = -\bar{\tilde{u}}_{cc}\bar{C}/\bar{\tilde{u}}_c$, $\bar{c}_t = \tilde{\sigma}\bar{\tilde{u}}_{c\xi}\xi_{c,t}/\bar{\tilde{u}}_c$, $\bar{\sigma}_1^{-1} \equiv -[\bar{\tilde{u}}_{ccc}\bar{C}/\bar{\tilde{u}}_{cc}+1]s_C^{-1}$, $\bar{\sigma}_2^{-1} \equiv -\bar{\tilde{u}}_{cc\xi}\bar{Y}/\bar{\tilde{u}}_{c\xi}$. Substituting $Y_t - G_t$ for C_t and defining $u(Y;\xi) \equiv \tilde{u}(Y-G;\xi_c)$, we obtain that $\sigma^{-1} \equiv -\bar{u}_{yy}\bar{Y}/\bar{u}_y = \tilde{\sigma}^{-1}s_C^{-1}$, $\sigma_1^{-1} \equiv -\bar{u}_{yyy}\bar{Y}^2/\bar{u}_y = -\sigma^{-1}(s_C^{-1} + \bar{\sigma}_1^{-1})$ and that $g_t \equiv \sigma\bar{u}_{y\xi}\xi_t/\bar{u}_y = (\hat{G}_t + s_C\bar{c}_t) + \mathcal{O}(||\xi||^2)$, $g_{2,t} \equiv \sigma_2\bar{u}_{yy\xi}\xi_t\bar{Y}/\bar{u}_y = -\sigma_2\sigma^{-1}s_C^{-1}\hat{G}_t - \sigma_2\sigma^{-1}\bar{\sigma}_2^{-1}g_t + \sigma_2\sigma^{-1}(\bar{\sigma}_2^{-1} - \bar{\sigma}_1^{-1})\hat{G}_t + \mathcal{O}(||\xi||^2)$.

Similarly, we introduce the notation $\nu \equiv \bar{\tilde{v}}_{hh}\bar{h}/\bar{\tilde{v}}_h$, $\nu_1 \equiv \bar{\tilde{v}}_{hhh}\bar{h}^2/\bar{\tilde{v}}_h$, $\nu_2 \equiv \bar{\tilde{v}}_{hh\xi}\bar{h}/\bar{\tilde{v}}_{h\xi}$, $\bar{h}_t = -\nu^{-1}\bar{\tilde{v}}_{h\xi}\xi_{h,t}/\bar{\tilde{v}}_h$. For the production function, we define $\phi^{-1} \equiv \bar{f}'\bar{h}/\bar{f}$, $\phi_1 \equiv \bar{f}''\bar{h}/\bar{f}'$, $\phi_2 \equiv \bar{f}'''\bar{h}^2/\bar{f}'$. Defining $v(y;\xi) \equiv \tilde{v}(f^{-1}(y/A);\xi)$, we obtain that $\omega \equiv \bar{v}_{yy}\bar{Y}/\bar{v}_y = \nu\phi - \phi\phi_1$, $q_t \equiv -\omega^{-1}\bar{v}_{y\xi}\xi_t/\bar{v}_y = \omega^{-1}(\nu\bar{h}_t + \phi(1+\nu)a_t) + \mathcal{O}(||\xi||^2)$. By defining $\bar{\omega}_1 \equiv (\nu_1\phi^2 - 3\nu\phi^2\phi_1 - \phi^2\phi_2 + 3\phi^2\phi_1^2 + \omega)/\omega$ we can further obtain that $\tilde{\omega}_1 \equiv \bar{v}_{yyy}\bar{Y}^2/\bar{v}_y = \omega(\bar{\omega}_1 - 1)$ and that $q_{2,t} \equiv \tilde{\omega}_2^{-1}\bar{v}_{yy\xi}\xi_t\bar{Y}/\bar{v}_y = \tilde{\omega}_2^{-1}(\omega^2 q_t - \tilde{\omega}_3 a_t + \tilde{\omega}_4\bar{h}_t) + \mathcal{O}(||\xi||^2)$ where we have defined $\omega_3 \equiv \omega(1+\omega) - 2\phi(\nu - \phi_1) - \omega_1$, $\omega_4 \equiv \nu\phi(\nu - \nu_2)$.

4.1 A Generalized Distribution of Intervals between Price Changes

Here we introduce more flexible notation for the frequency distribution of the intervals between price changes; among other cases, our general model includes both classic Taylor contracts (price commitments for a fixed period of time) and the Calvo model as special cases.[40] Let γ_k be the fraction of the industries in any period that made their last price adjustment k periods earlier, for $k = 0, 1, 2, \ldots$. We assume that $\gamma_{k+1} \leq \gamma_k$ for each k, and that all γ_k are non-negative and sum to 1. We furthermore assume that a price commitment already in effect for k periods will continue in effect for another period with probability γ_{k+1}/γ_k, and that new price commitments are chosen with a knowledge only of the probability distribution of possible times until revision of the price, not the actual length of time that a given commitment will continue in effect (except when all commitments last for exactly n periods, as in the Taylor model). In this notation, the Taylor model corresponds to the special case in which $\gamma_k = 1/n$ for each $0 \leq k \leq n-1$ and $\gamma_k = 0$ for each $k \geq n$. The Calvo model considered earlier corresponds to the case in which $\gamma_k = (1-\alpha)\alpha^k$ for some $0 < \alpha < 1$.

[40] The notation introduced here also allows us to analyze more complex cases, such as the one assumed in Khan *et al.* (2003).

Households maximize their level of expected utility

$$U_t = E_{t_0}\beta^{t-t_0}\tilde{U}_t \tag{4.5}$$

where the utility flows can be written as

$$\begin{aligned}\tilde{U}_t &= u(Y_t;\xi_t) - \int_0^1 v(y_t(j);\xi_t)dj \\ &= u(Y_t;\xi_t) - \sum_{k=0}^{\infty}\gamma_k v(Y_t z(p^*_{t-k}/P_t);\xi_t). \end{aligned} \tag{4.6}$$

Here p^*_t is the price chosen in period t by each industry that adjusts its prices at that date, and P_t is a homogeneous-degree-one aggregate of the industry prices $\{p^*_{t-k}\}$, implicitly defined by (4.2) which can be written as

$$\sum_{k=0}^{\infty}\gamma_k h(p^*_{t-k}/P_t) = 1. \tag{4.7}$$

Note that past prices $\{p^*_{t_0-k}\}$ remain relevant for the evaluation of welfare looking forward from period t_0, even if they do not constrain the possible equilibrium evolution of prices and output from t_0 onward. In fact only the vector of (previous period) relative prices D_{t_0-1}, defined by

$$D_{jt} = p^*_{t-j}/P_t$$

for $j \geq 0$, identifies predetermined state variables that matter for defining the feasible level of utility from t_0 onward. Given the relative price $\tilde{p}_t \equiv p^*_t/P_t$ chosen by firms that adjust their prices in period t, the implied aggregate inflation rate $\Pi_t \equiv P_t/P_{t-1}$ is given by

$$\Pi_t = \Pi(\tilde{p}_t, D_{t-1}),$$

where the function $\Pi(\cdot)$ is implicitly defined by

$$\gamma_0 h(\tilde{p}_t) + \sum_{k=0}^{\infty}\gamma_{k+1}h(D_{k,t-1}/\Pi_t) = 1.$$

Using this, we can write a law of motion for the vector of predetermined states,

$$D_t = T(\tilde{p}_t, D_{t-1}) \tag{4.8}$$

in which

$$T_0(\tilde{p}, D) \equiv \tilde{p},$$

and

$$T_j(\tilde{p}, D) \equiv D_{j-1}/\Pi(\tilde{p}, D)$$

for each $j \geq 1$. We can furthermore write

$$\tilde{U}_t = U(Y_t, D_t, \xi_t) \tag{4.9}$$

where $U(\cdot)$ is defined by the right-hand side of (4.6). It then follows from (4.8) that the utility flow from t_0 onward depends on the evolution of the endogenous variables $\{\tilde{p}_t, Y_t\}$ and the exogenous disturbances $\{\xi_t\}$ from t_0 onward, together with the vector of initial states D_{t_0-1}.

As in section xx, each of the suppliers that revise their prices in period t choose the same new price p_t^*. The probability that a new price adopted in period t still applies in period $t+k$, for each $k \geq 0$, must equal γ_k/γ_0. It follows that the first-order condition for the optimal choice of prices must satisfy

$$E_t \sum_{k=0}^{\infty} \gamma_k Q_{t,t+k} \tilde{g}(p_t^*/P_{t+k}; Y_{t+k}, \xi_{t+k}) = 0, \tag{4.10}$$

for each $t \geq t_0$, where the function $\tilde{g}(p^*/P; Y, \xi) \equiv \Pi_1(p^*, p^j, P; Y, \xi)$. In terms of the above notation for relative prices we can write (4.10) as

$$E_t \sum_{k=0}^{\infty} \beta^k \gamma_k g(D_{k,t+k}; Y_{t+k}, \xi_{t+k}) = 0, \tag{4.11}$$

where the function $g(\cdot)$ is defined by

$$g(D_j, Y; \xi) \equiv u_y(Y; \xi) z(D_j) Y M_j(D_j, Y; \xi),$$

where

$$M_j(D_j, Y; \xi) \equiv (1 - \tau_t) D_j (1 - \theta(D_j)) + \mu^w \theta(D_j) \frac{v_y(Y z(D_j), \xi)}{u_y(Y; \xi)},$$

while $\theta(D_j) \equiv -D_j z'(D_j)/z(D_j)$. The restriction (4.11) together with (4.8) constraints the feasible evolution of the variables $\{\tilde{p}_t, Y_t, D_t\}$, given the initial condition D_{t_0-1}.

Let $x_t \equiv (Y_t,\, \tilde{p}_t,\, D_t)$ and define

$$F_{j,t} \equiv E_t \sum_{k=j}^{\infty} \beta^{k-j} \gamma_k g(D_{k,t+k-j}; Y_{t+k-j}, \xi_{t+k-j}) \tag{4.12}$$

for each $j \geq 1$. Let F_t be the vector the j the element of which is $F_{j,t}$, for $j \geq 1$. In general, this is an infinite-dimensional vector. In the case that $\gamma_k = 0$ for all $k \geq n$, some finite n, then only the first $n-1$ elements of F_t matter.

Let $\mathcal{F}$ be the set of values for $(F_t,\, D_{t-1})$ such that there exist paths $\{x_T\}$ for dates $T \geq t$ that satisfy (4.8) and (4.11) for each T, that are consistent with the specified elements of F_t and that imply a well-defined value for the objective U_t defined in (4.5). For any $(F_t,\, D_{t-1}) \in \mathcal{F}$, let $V(D_{t-1}, F_t; \xi_t)$ the maximum attainable value of U_t among the state-contingent paths that satisfy the constraints just mentioned. The t_0–optimal plan can be obtained as in section xx as a two-stage optimization problem. Let us focus on the second stage. This second stage is of the same form as the Ramsey problem itself, except that there are additional constraints associated with precommitted values of the elements $F_{t+1}(\xi_{t+1})$. Let us consider this problem looking forward from period t_0 under the additional set of constraints $F_{t_0} = \bar{F}_{t_0}$ (that may depend on the exogenous state ξ_{t_0}). The constrained policy problem is then to choose state-contingent paths $\{x_t\}$ from t_0 onward, that satisfy the constraints (4.8) and (4.11) for all $t \geq t_0$, and the vector of constraints $F_{t_0} = \bar{F}_{t_0}$, (where F_{j,t_0} is given by (4.12)) in order to maximize (4.5) (where $\tilde{U}_t$ is given by (4.9)), given the initial state D_{t_0-1} and precommitted values $\bar{F}_{t_0}$. This constraint problem can be as well written in a sequential way. Let us define the following functional relationships

$$\hat{J}[x, F(\cdot)](\xi_t) \equiv U(Y_t, D_t; \xi_t) + \beta E_t V(D_t, F_{t+1}; \xi_{t+1})$$

$$\hat{F}_0[x, F(\cdot)](\xi_t) = \gamma_0 g(\tilde{p}_t; Y_t, \xi_t) + \beta E_t F_{1,t+1},$$

$$\hat{F}_j[x, F(\cdot)](\xi_t) \equiv \gamma_j g(D_{jt}; Y_t, \xi_t) + \beta E_t F_{j+1,t+1}$$

for each $j \geq 1$.

Then in each period $t \geq t_0$, the monetary authority chooses x_t, and state-contingent commitments $F_{t+1}(\xi_{t+1})$ so as to maximize $\hat{J}[x, F(\cdot)](\xi_t)$ over values of x_t and $F_{t+1}(\xi_{t+1})$ such that

(i) D_t and $\tilde{p}_t$ satisfy (4.8);

(ii) $\hat{F}_0[x, F(\cdot)](\xi_t) = 0$ and $\hat{F}_j[x, F(\cdot)](\xi_t) = F_{j,t}$ for each $j \geq 1$.

(iii) $(F_{t+1},\, D_t) \in \mathcal{F}(\xi_{t+1})$

given the predetermined states D_{t-1}, the exogenous states ξ_t, and the precommitted values $F_t = \bar{F}_t$.

In the appendix we prove the following proposition

PROPOSITION 7. Given some $(D_{t_0-1},\, F_{t_0}) \in \mathcal{F}$ consider the sequential decision problem in which each period $t \geq t_0$, $(x_t, F_{t+1}(\cdot))$ are chosen to maximize $\hat{J}[x, F(\cdot)](\xi_t)$, subject to constraints $(i) - (iii)$ stated above, given the predetermined state variable D_{t-1} and the precommitted values F_t. Then the process $\{x_t\}$ that is chosen in this way is the process that maximizes U_{t_0} among all of the paths consistent with (4.8) and (4.11) for each $t \geq t_0$, given D_{t_0-1} and also consistent with the specified values F_{t_0}.

As in section xx, our aim is to characterize policy that solves this constrained optimization problem for initial conditions near certain steady-state values. We first show the existence of a steady state, *i.e.,* of an optimal policy (under appropriate initial conditions) that involves constant values of all variables. To this end we consider the purely deterministic case, in which the exogenous disturbances $\xi_{c,t}$,G_t, $\xi_{h,t}$,A_t,μ_t^w,τ_t each take constant values $\xi_c = \xi_h = 0$ (without loss of generality), $\bar{\tau}, \bar{A}, \bar{\mu}^w > 0$, $\bar{G} \geq 0$ for all $t \geq t_0$. We wish to find an initial degree of price dispersion D_{t_0-1} and initial commitments $F_{t_0} = \bar{F}$ such that the solution to the problem defined above involves a constant policy $x_t = \bar{x}$, $F_{t+1} = \bar{F}$ each period in which $\bar{D}$ is equal to the initial price dispersion. We show in the appendix that the first-order conditions for this problem admit a steady-state solution of this form, and we verify below under which conditions the second-order conditions for a local optimum are also satisfied.

We show that the result of section xx generalizes to this more general specification of the policy problem and most important to other price-setting mechanisms than the Calvo's model. Indeed, we find that $\bar{\Pi} = 1$(zero inflation), and correspondingly that $\bar{D} = 1$(zero price dispersion) is a steady solution.

We next wish to characterize the optimal responses to small perturbations of the initial conditions and small fluctuations in the disturbance processes around the above values. To do this, we compute a linear-quadratic approximate problem, the solution to which represents a linear approximation to the solution to the "stage two" policy problem.

We begin by computing a Taylor-series approximation to our welfare measure (4.5), expanding around the steady-state allocation defined above, in which $y_t(i) = \bar{Y}$ for each good at all times and $\xi_t = 0$ at all times. As a second-order (logarithmic) approximation to this measure, we obtain[41]

$$\begin{aligned} U_{t_0} &= \bar{u}_y \bar{Y} \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ \Phi \hat{Y}_t - \frac{1}{2} u_{yy} \hat{Y}_t^2 + \hat{Y}_t u_{y\xi} \xi_t - \frac{1}{2} u_d E_j \hat{D}_{j,t}^2 \} \\ &\quad + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned} \tag{4.13}$$

where u_{yy} and $u_{y\xi}\xi_t$ are the same as in section xx and u_d is defined in the appendix. As before, the coefficient

$$\Phi \equiv 1 - \frac{\bar{\theta} - 1}{\bar{\theta}} \frac{1 - \bar{\tau}}{\bar{\mu}^w} < 1$$

measure the inefficiency of the steady-state output level $\bar{Y}$ where now $\bar{\theta} = \theta(1)$. We have further defined

$$E_j \hat{D}_{j,t}^2 \equiv \sum_{j=0}^{\infty} \gamma_j \hat{D}_{j,t}^2,$$

as an index of second moment price dispersion. To solve for the linear term in (4.13) we consider a second-order approximation to the generic constraint

$$\gamma_j g(D_{j,t}, Y_t; \xi_t) + \beta F_{j+1,t+1} = F_{j,t} \tag{4.14}$$

for each j where $F_{0,t} = 0$ obtaining

$$\gamma_j M_{j,t} + \gamma_j [(1 - \sigma^{-1}) \hat{Y}_t + \sigma^{-1} g_t + \bar{z}' \hat{D}_{j,t}] M_{j,t} = \bar{u}_y^{-1} \bar{Y}^{-1} (F_{j,t} - \beta F_{j+1,t+1}) + \mathcal{O}(||\xi||^3), \tag{4.15}$$

where $M_{j,t}$ is a short-hand writing of the function $M_{j,t}(\cdot)$ and $\bar{z}'$ denote the first-derivative of the function $z(\cdot)$ evaluated at the steady-state.

We now sum (4.15) across j and obtain

$$\begin{aligned} \bar{u}_c^{-1} \bar{Y}^{-1} (\tilde{F}_t - \beta \tilde{F}_{t+1}) &= E_j M_{j,t} + [(1 - \sigma^{-1}) \hat{Y}_t + \sigma^{-1} g_t] E_j M_{j,t} \\ &\quad + \bar{z}' E_j \{ \hat{D}_{j,t} M_{j,t} \} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{4.16}$$

where we have defined $\tilde{F}_t \equiv \sum_{j=1}^{\infty} F_{j,t}$. As shown in the appendix, we can further take a second-order approximation to $M_{j,t}$ and substitute in (4.16). After integrating

[41] See the appendix for details.

forward the resulting expression from period t_0, we obtain that

$$\begin{aligned} b_f \tilde{F}_{t_0} &= \bar{u}_y \bar{Y} \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ \hat{Y}_t + \frac{1}{2} b_d E_j \hat{D}_{j,t}^2 + \frac{1}{2} b_{yy} \hat{Y}_t^2 - \hat{Y}_t b_{y\xi} \xi_t \} \\ & \quad + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{4.17}$$

for certain coefficients defined in the appendix. We can then use (4.17) to write the discounted sum of output terms in (4.13) as a function of purely quadratic terms, up to a residual of third order. We can then obtain

$$U_{t_0} = -\Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ \frac{q_y}{2} (\hat{Y}_t - \hat{Y}_t^*)^2 + \frac{q_d}{2} E_j \hat{D}_{j,t}^2 \} + \Phi b_f \tilde{F}_{t_0} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \tag{4.18}$$

where we have defined $\Omega \equiv \bar{u}_y \bar{Y}$, $q_y \equiv u_{yy} + \Phi b_{yy}$, $q_d \equiv u_d + \Phi b_d$ and $\hat{Y}_t^* \equiv q_y^{-1}(u_{y\xi}\xi_t + \Phi b_{y\xi}\xi_t)$. Once again we are interested in characterizing optimal policy from a timeless perspective. We observe that $\tilde{F}_{t_0}$ is a function of the additional commitments at time 0 and then it is a predetermined variable and independent of policy. Hence, over the set of admissible policies, higher values of (4.18) correspond to lower values of

$$L_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ \frac{q_y}{2} (\hat{Y}_t - \hat{Y}_t^*)^2 + \frac{q_d}{2} E_j \hat{D}_{j,t}^2 \}. \tag{4.19}$$

Because this loss function is purely quadratic, it is possible to evaluate it to second order using only a first-order approximation to the equilibrium evolution of output and price dispersion. A log-linear approximation of the constraint (4.11) at each date t yields to

$$E_t \sum_{k=0}^{\infty} \beta^k \gamma_k \hat{D}_{k,t+k} = E_t \sum_{k=0}^{\infty} \beta^k \gamma_k \lambda_y (\hat{Y}_{t+k} - \hat{Y}_{t+k}^n) \tag{4.20}$$

where we have defined $\lambda_y \equiv (\omega + \sigma^{-1})/(1 + \varsigma)$ with $\varsigma \equiv \bar{\theta}^{-1}(\bar{\theta} - 1)^{-1}\bar{\theta}' - \omega \bar{z}'$ and $\bar{\theta}'$, $\bar{z}'$ are the derivative of the functions $\theta(\cdot)$, $z(\cdot)$ at the steady state. [42] We further note that a log-linear approximation to (4.7) implies

$$\sum_{j=0}^{\infty} \gamma_j \hat{D}_{j,t} = 0. \tag{4.21}$$

[42]Equation (4.20) is implied by (4.15) for $j = 0$ since $F_{j,t} = 0$.

Moreover $\hat{D}_{j,t} = \hat{D}_{j-1,t-1} - \pi_t$ for each $j \geq 1$. It then follows that

$$\gamma_0 \hat{D}_{0,t} + \sum_{j=0}^{\infty} \gamma_{j+1}(\hat{D}_{j,t-1} - \pi_t) = 0,$$

which implies that

$$\pi_t = \frac{\gamma_0}{1-\gamma_0}\hat{D}_{0,t} + \sum_{j=0}^{\infty} \frac{\gamma_{j+1}}{1-\gamma_0}\hat{D}_{j,t-1}. \tag{4.22}$$

We can then substitute into $\hat{D}_{j,t} = \hat{D}_{j-1,t-1} - \pi_t$ and obtain

$$\hat{D}_{j,t} = \hat{D}_{j-1,t-1} - \frac{\gamma_0}{1-\gamma_0}\hat{D}_{0,t} - \sum_{j=0}^{\infty} \frac{\gamma_{j+1}}{1-\gamma_0}\hat{D}_{j,t-1} \tag{4.23}$$

for each $j \geq 1$. To complete to characterize the optimal policy problem from a timeless perspective, we need to discuss the nature of the additional constraints at time t_0 on F_{t_0}. Indeed, the log-linear terms of (4.15) imply

$$b_m \bar{u}_y^{-1} \bar{Y}^{-1} F_{j,t} = E_t \sum_{k=0}^{\infty} \beta^k \gamma_{j+k} \lambda_y (\hat{Y}_{t+k} - \hat{Y}_{t+k}^n) - \sum_{k=0}^{\infty} \beta^k \gamma_{j+k} \hat{D}_{j+k,t+k}, \tag{4.24}$$

for each $j \geq 1$ where b_m is defined in the appendix. A restriction on $F_{j,t}$ implies a restriction on the possible future evolution of the variables of the RHS of (4.24). An optimal policy from a timeless perspective is a policy from date t_0 onward that minimizes the quadratic loss function (4.19) subject to the constraints implied by the linear structural relations (4.20) and (4.23) holding in each period $t \geq t_0$ and to the constraints that certain predetermined values for F_{j,t_0} (as in (4.24) for each $j \geq 1$ be achieved, given the initial conditions $\hat{D}_{j,t_0-1}$ for each $j \geq 0$.

4.2 The Calvo Model with General Preferences and Technology

We can simplify further the analysis if we again assume the Calvo frequency distribution for the intervals between price changes, $\gamma_k = (1-\alpha)\alpha^k$. Under this assumption, (4.21) up to first-order terms implies that

$$\sum_{j=0}^{\infty} (1-\alpha)\alpha^j \hat{D}_{j,t} = 0$$

and that

$$(1-\alpha)\hat{D}_{0,t} + \sum_{j=1}^{\infty}(1-\alpha)\alpha^j(\hat{D}_{j-1,t-1} - \pi_t) = 0.$$

It then follows that

$$(1-\alpha)\hat{D}_{0,t} + \alpha\sum_{j=0}^{\infty}(1-\alpha)\alpha^j\hat{D}_{j,t-1} - \alpha\pi = 0,$$

and then that

$$\hat{D}_{0,t} = \frac{\alpha}{(1-\alpha)}\pi_t. \tag{4.25}$$

We can further note that we can simplify $E_j^2\hat{D}_{j,t}$ as

$$\begin{aligned} E_j^2\hat{D}_{j,t} &= (1-\alpha)\hat{D}_{0,t}^2 + \sum_{j=1}^{\infty}(1-\alpha)\alpha^j(\hat{D}_{j-1,t-1} - \pi_t)^2 \\ &= (1-\alpha)\hat{D}_{0,t}^2 + \alpha\sum_{j=0}^{\infty}(1-\alpha)\alpha^j(\hat{D}_{j,t-1}^2 - 2\hat{D}_{j,t-1}\pi_t + \pi_t^2) \\ &= \frac{\alpha^2}{(1-\alpha)}\pi_t^2 + \alpha E_j\hat{D}_{j,t-1}^2 + \alpha\pi_t^2 + \mathcal{O}(||\xi||^3) \\ &= \frac{\alpha}{(1-\alpha)}\pi_t^2 + \alpha E_j\hat{D}_{j,t-1}^2 + \mathcal{O}(||\xi||^3), \end{aligned}$$

where we have used (4.25). Integrating from t_0 we finally obtain

$$\sum_{t=t_0}^{\infty}\beta^{t-t_0}E_j\hat{D}_{j,t}^2 = \sum_{t=t_0}^{\infty}\beta^{t-t_0}\frac{\alpha}{(1-\alpha)(1-\alpha\beta)}\frac{\pi_t^2}{2} + \text{t.i.p} + \mathcal{O}(||\xi||^3). \tag{4.26}$$

Substituting (4.26) into (4.19) we obtain that

$$L_{t_0} = E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{\frac{q_y}{2}(\hat{Y}_t - \hat{Y}_t^*)^2 + \frac{q_\pi}{2}\pi_t^2\} + \text{t.i.p} + \mathcal{O}(||\xi||^3) \tag{4.27}$$

where we have defined $q_\pi \equiv \frac{\alpha}{(1-\alpha)(1-\alpha\beta)}q_d$.

Equation (4.27) can be evaluated by the log-linear approximation to the structural equations (4.20) and (4.23) which can be written together to imply

$$\pi_t = \kappa(\hat{Y}_t - \hat{Y}_t^n) + \beta E_t\pi_{t+1}, \tag{4.28}$$

which is the same "New Keynesian" aggregate supply equation of section xx except that now $\kappa = (1-\alpha)(1-\alpha\beta)(\omega+\sigma^{-1})/(\alpha(1+\varsigma))$.

Moreover equation (4.20) at time t_0 together with (4.24) imply that

$$\frac{1-\alpha\beta}{(1-\alpha)\alpha^j}(\bar{\theta}-1)^{-1}\bar{u}_y^{-1}\bar{Y}^{-1}F_{j,t_0} = \hat{D}_{0,t_0} - \hat{D}_{j,t_0}$$

for each $j \geq 1$. Multiplying by γ_j and summing across j, we obtain that

$$\begin{aligned}(\bar{\theta}-1)^{-1}\bar{u}_y^{-1}\bar{Y}^{-1}\tilde{F}_{t_0} &= \frac{\hat{D}_{0,t_0}}{(1-\alpha\beta)} \\ &= \frac{\alpha}{(1-\alpha)(1-\alpha\beta)}\pi_{t_0}\end{aligned}$$

where we have used (4.25) from which it follows that a commitment on the variables F_j at time t_0 and on $\tilde{F}_{t_0}$ implies a commitment on π_{t_0}. As in section xx, a commitment on π_{t_0} represents the only commitment regarding future outcome that it is needed to characterize optimal policy from a timeless perspective in the LQ optimization problem. An optimal policy from a timeless perspective is a policy from date t_0 onward that minimizes the quadratic loss function (4.27) subject to the constraints implied by the linear structural relation (4.28) holding in each period $t \geq t_0$ and subject also to the constraints that certain predetermined values for π_{t_0} be achieved.

As in section xx, we are interested in studying under which conditions the process $\{\pi_t, \hat{Y}_t\}$ that satisfy the first-order conditions for the LQ optimization problem represent a loss minimum. In particular proposition 3 still applies in his case. However, the parameters κ, q_y, q_π are now different. If we restrict attention to Dixit-Stiglizt preferences, we obtain that κ is equal to that of section xx, and now q_y and q_π are such that

$$q_y \equiv (\omega+\sigma^{-1}) + \Phi(1-\sigma^{-1}) + \Phi\frac{\sigma^{-1}(1-s_C^{-1}) + \omega(\bar{\omega}_1-\omega) - \sigma^{-1}(\bar{\sigma}_1^{-1}-\sigma^{-1})}{(\omega+\sigma^{-1})},$$

$$q_\pi \equiv \frac{\theta}{\kappa}[(\omega+\sigma^{-1}) + \Phi(1-\sigma^{-1})] + \frac{\theta}{\kappa}\frac{\Phi\omega(\bar{\omega}_1-\omega)\theta}{(1+\theta\omega)}.$$

It is now the case that it is not necessary to assume an implausible large share of government purchases in total demand to obtain that randomization is welfare improving. Indeed, Dupor (2003) has shown that it is sufficient to have a high value of the relative prudence of the consumers, i.e. $-\bar{\tilde{U}}_{ccc}\bar{C}/\bar{\tilde{U}}_{cc}$. (In our case the parameter $\bar{\sigma}_1^{-1}+1$.) He assume a model in which prices are fixed one-period in advance, so that there is no cost of price dispersion, in our case q_π should be assumed $q_\pi = 0$. Dupor

(2003) has further assumed zero government purchases (which implies $s_G = 0$, and $s_C = 1$), zero tax rates and wage mark up ($\bar{\tau} = 0$ and $\bar{\mu}^w = 1$) and linear disutility of producing the goods ($\omega = \bar{\omega}_1 = 0$). In this case, randomization can be welfare improving if and only if $q_y \leq 0$ which requires

$$q_y = \sigma^{-1} + \Phi - \Phi(\bar{\sigma}_1^{-1} - \sigma^{-1}) \leq 0$$

from which it follows that

$$\bar{\sigma}_1^{-1} - 1 \geq \theta\sigma^{-1}$$

since $\Phi = \theta^{-1}$ under this specification.[43] It is then needed that the relative prudence parameter be high enough with respect to the risk-aversion coefficient and at the same time that the degree of market-competition is not high. In our more general model, the cases for randomization can be reinforced when also $\bar{\omega}_1$ is low with respect to ω, when s_G is high and for high values of Φ which can depends on high values of the steady-state distortionary taxes or wage mark-up.

This analysis shows instead that proposition 4 and 5 hold crucially for the assumption of isoelastic preferences. Indeed, by continuing to assume Dixit-Stiglitz aggregators, we obtain that now

$$\hat{Y}_t^* = \omega_1 \hat{Y}_t^n - \omega_2 \hat{G}_t + \omega_3 \hat{u}_t^w + \omega_4 \hat{\tau}_t + \omega_5 g_t + \omega_6 a_t + \omega_7 \bar{h}_t + \mathcal{O}(||\xi||^2)$$

where

$$\omega_1 = q_y^{-1}[(\omega + \sigma^{-1}) + \Phi_y(1 - \sigma^{-1})], \qquad \omega_2 = \frac{\Phi}{q_y(\omega + \sigma^{-1})}[\sigma^{-1} s_C^{-1} - \sigma^{-1}(\bar{\sigma}_2^{-1} - \bar{\sigma}_1^{-1})],$$

$$\omega_3 \equiv q_y^{-1}(1 - \Phi), \qquad \omega_4 \equiv q_y^{-1}\omega_\tau, \qquad \omega_5 = \frac{\Phi}{q_y(\omega + \sigma^{-1})}\sigma^{-1}(\sigma^{-1} - \bar{\sigma}_2^{-1}),$$

$$\omega_6 = -\frac{\Phi}{q_y(\omega + \sigma^{-1})}\tilde{\omega}_3, \qquad \omega_7 = \frac{\Phi}{q_y(\omega + \sigma^{-1})}\tilde{\omega}_4.$$

We can then re-write the aggregate supply relation (4.28) as

$$\pi_t = \kappa x_t + u_t + \beta E_t \pi_{t+1}$$

where the composite "cost-push" shock is now

$$u_t = \kappa[(\omega_1 - 1)\hat{Y}_t^n - \omega_2 \hat{G}_t + \omega_3 \hat{u}_t^w + \omega_4 \hat{\tau}_t + \omega_5 g_t + \omega_6 a_t + \omega_7 \bar{h}_t]$$

[43] This result indeed coincide with Dupor's result, once we define in his notation $\rho = \sigma^{-1}$ and $\eta = 1 + \bar{\sigma}_1^{-1}$.

for the same $\tilde{q}_1$ as in the previous section. It follows now that isoelastic preferences are a necessary condition, for the policy of price stability to minimize the loss function. In particular now it is important to distinguish between shocks to the disutility of working and productivity shocks in determining the decomposition of the composite cost-push shock.

A Appendix

A.1 The deterministic steady state

Here we show the existence of a steady state, *i.e.,* of an optimal policy (under appropriate initial conditions) of the ‘recursive policy problem defined in Proposition 2 that involves constant values of all variables. We consider a deterministic problem in which the exogenous disturbances $\bar{C}_t$, G_t, $\bar{H}_t$, A_t, μ_t^w, τ_t each take constant values $\bar{C}$, $\bar{H}$, $\bar{A}$, $\bar{\mu}^w$, $\bar{\tau} > 0$ and $\bar{G} \geq 0$ for all $t \geq t_0$. We wish to find an initial degree of price dispersion Δ_{t_0-1} and initial commitments $X_{t_0} = \bar{X}$ such that the recursive (or "stage two") problem involves a constant policy $x_{t_0} = \bar{x}$, $X_{t+1} = \bar{X}$ each period, in which $\bar{\Delta}$ is equal to the initial price dispersion.

We thus consider the problem of maximizing

$$U_{t_0} = \sum_{t=t_0}^{\infty} \beta^{t-t_0} U(Y_t, \Delta_t) \tag{A.29}$$

subject to the constraints

$$K_t p(\Pi_t)^{\frac{1+\omega\theta}{\theta-1}} = F_t, \tag{A.30}$$

$$F_t = (1-\tau_t) f(Y_t) + \alpha\beta \Pi_{t+1}^{\theta-1} F_{t+1}, \tag{A.31}$$

$$K_t = k(Y_t) + \alpha\beta \Pi_{t+1}^{\theta(1+\omega)} K_{t+1}, \tag{A.32}$$

$$\Delta_t = \alpha \Delta_{t-1} \Pi_t^{\theta(1+\omega)} + (1-\alpha) p(\Pi_t)^{-\frac{\theta(1+\omega)}{1-\theta}}, \tag{A.33}$$

and given the specified initial conditions Δ_{t_0-1}, X_{t_0}, where we have defined

$$p(\Pi_t) \equiv \left(\frac{1 - \alpha \Pi_t^{\theta-1}}{1-\alpha} \right).$$

We introduce Lagrange multipliers ϕ_{1t} through ϕ_{4t} corresponding to constraints (A.30) through (A.33) respectively. We also introduce multipliers dated t_0 corresponding to the constraints implied by the initial conditions $X_{t_0} = \bar{X}$; the latter multipliers are normalized in such a way that the first-order conditions take the same form at date t_0 as at all later dates. The first-order conditions of the maximization problem are then the following. The one with respect to Y_t is

$$U_y(Y_t, \Delta_t) - (1-\tau_t) f_y(Y_t) \phi_{2t} - k_y(Y_t) \phi_{3t} = 0; \tag{A.34}$$

that with respect to Δ_t is

$$U_\Delta(Y_t, \Delta_t) + \phi_{4t} - \alpha\beta\Pi_{t+1}^{\theta(1+\omega)}\phi_{4,t+1} = 0; \tag{A.35}$$

that with respect to $\Pi_t$is

$$\frac{1+\omega\theta}{\theta-1}p(\Pi_t)^{\frac{(1+\omega\theta)}{\theta-1}-1}p_\pi(\Pi_t)K_t\phi_{1,t} - \alpha(\theta-1)\Pi_t^{\theta-2}F_t\phi_{2,t-1}$$
$$-\theta(1+\omega)\alpha\Pi_t^{\theta(1+\omega)-1}K_t\phi_{3,t-1}+$$
$$-\theta(1+\omega)\alpha\Delta_{t-1}\Pi_t^{\theta(1+\omega)-1}\phi_{4t} - \frac{\theta(1+\omega)}{\theta-1}(1-\alpha)p(\Pi_t)^{\frac{(1+\omega\theta)}{\theta-1}}p_\pi(\Pi_t)\phi_{4t} = 0; \tag{A.36}$$

that with respect to F_t is

$$-\phi_{1t} + \phi_{2t} - \alpha\Pi_t^{\theta-1}\phi_{2,t-1} = 0; \tag{A.37}$$

that with respect to K_t is

$$p(\Pi_t)^{\frac{1+\omega\theta}{\theta-1}}\phi_{1t} + \phi_{3t} - \alpha\Pi_t^{\theta(1+\omega)}\phi_{3,t-1} = 0; \tag{A.38}$$

We search for a solution to these first-order conditions in which $\Pi_t = \bar{\Pi}$, $\Delta_t = \bar{\Delta}$, $Y_t = \bar{Y}$ at all times. A steady-state solution of this kind also requires that the Lagrange multipliers take constant values. We furthermore conjecture the existence of a solution in which $\bar{\Pi} = 1$, as stated in the text. Note that such a solution implies that $\bar{\Delta} = 1$, $p(\bar{\Pi}) = 1$, $p_\pi(\bar{\Pi}) = -(\theta-1)\alpha/(1-\alpha)$, and $\bar{K} = \bar{F}$. Using these substitutions, we find that (the steady-state version of) each of the first-order conditions (A.34) – (A.38) is satisfied if the steady-state values satisfy

$$[(1-\bar{\tau})f_y(\bar{Y}) - k_y(\bar{Y})]\phi_2 = U_y(\bar{Y}, 1),$$
$$(1-\alpha\beta)\phi_4 = -U_\Delta(\bar{Y}, 1),$$
$$\phi_1 = (1-\alpha)\phi_2,$$
$$\phi_3 = -\phi_2.$$

These equations can obviously be solved (uniquely) for the steady-state multipliers, given any value $\bar{Y} > 0$.

Similarly, (the steady-state versions of) the constraints (A.30) – (A.33) are satisfied if

$$(1-\bar{\tau})u_c(\bar{Y} - \bar{G}) = \frac{\theta}{\theta-1}\bar{\mu}^w v_y(\bar{Y}), \tag{A.39}$$
$$\bar{K} = \bar{F} = (1-\alpha\beta)^{-1}k(\bar{Y}),$$

Equation (A.39) can be solved for the steady-state value $\bar{Y}$.

A.2 A second-order approximation to utility (equations (2.1) and (2.3))

We derive here equations (2.1) and (2.3) in the main text, taking a second-order approximation to (equation (1.8)) following the treatment in Woodford (2003, chapter 6). We start by approximating the expected discounted value of the utility of the representative household

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[u(Y_t; \xi_t) - \int_0^1 v(y_t(i); \xi_t) di \right]. \tag{A.40}$$

First we note that

$$\int_0^1 v(y_t(i); \xi_t) di = \frac{\lambda}{1+\nu} \frac{Y_t^{1+\omega}}{A_t^{1+\omega} \bar{H}_t^{\nu}} \Delta_t = v(Y_t; \xi_t) \Delta_t$$

where Δ_t is the measure of price dispersion defined in the text. We can then write (A.40) as

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[u(Y_t; \xi_t) - v(Y_t; \xi_t) \Delta_t \right]. \tag{A.41}$$

The first term in (A.41) can be approximated using a second-order Taylor expansion around the steady state defined in the previous section as

$$\begin{aligned}
u(Y_t; \xi_t) &= \bar{u} + \bar{u}_c \tilde{Y}_t + \bar{u}_\xi \xi_t + \frac{1}{2} \bar{u}_{cc} \tilde{Y}_t^2 + \bar{u}_{c\xi} \xi_t \tilde{Y}_t + \frac{1}{2} \xi_t' \bar{u}_{\xi\xi} \xi_t + \mathcal{O}(||\xi||^3) \\
&= \bar{u} + \bar{Y} \bar{u}_c \cdot (\hat{Y}_t + \frac{1}{2} \hat{Y}_t^2) + \bar{u}_\xi \xi_t + \frac{1}{2} \bar{Y} \bar{u}_{cc} \hat{Y}_t^2 + \\
&\qquad + \bar{Y} \bar{u}_{c\xi} \xi_t \hat{Y}_t + \frac{1}{2} \xi_t' \bar{u}_{\xi\xi} \xi_t + \mathcal{O}(||\xi||^3) \\
&= \bar{Y} u_c \hat{Y}_t + \frac{1}{2} [\bar{Y} \bar{u}_c + \bar{Y}^2 \bar{u}_{cc}] \hat{Y}_t^2 - \bar{Y}^2 \bar{u}_{cc} g_t \hat{Y}_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= \bar{Y} \bar{u}_c \left\{ \hat{Y}_t + \frac{1}{2} (1 - \sigma^{-1}) \hat{Y}_t^2 + \sigma^{-1} g_t \hat{Y}_t \right\} + \\
&\qquad + \text{t.i.p.} + \mathcal{O}(||\xi||^3),
\end{aligned} \tag{A.42}$$

where a bar denotes the steady-state value for each variable, a tilde denotes the deviation of the variable from its steady-state value (*e.g.*, $\tilde{Y}_t \equiv Y_t - \bar{Y}$), and a hat refers to the log deviation of the variable from its steady-state value (*e.g.*, $\hat{Y}_t \equiv \ln Y_t / \bar{Y}$). We use ξ_t to refer to the entire vector of exogenous shocks,

$$\xi_t' \equiv \left[\begin{array}{ccccc} \hat{G} & g_t & q_t & \hat{\mu}_t^w & \hat{\tau}_t \end{array} \right],$$

in which $\hat{G}_t \equiv (G_t - \bar{G})/\overline{Y}$, $g_t \equiv \hat{G}_t + s_C \bar{c}_t$, $\omega q_t \equiv \nu \bar{h}_t + \phi(1+\nu) a_t$, $\hat{\mu}_t^w \equiv \ln \mu_t^w / \bar{\mu}^w$, $\hat{\tau}_t \equiv (\tau_t - \bar{\tau})/\bar{\tau}$, $\bar{c}_t \equiv \ln \bar{C}_t/\bar{C}$, $a_t \equiv \ln A_t/\bar{A}$, $\bar{h}_t \equiv \ln \bar{H}_t/\bar{H}$. Moreover, we use the definitions $\sigma^{-1} \equiv \tilde{\sigma}^{-1} s_C^{-1}$with $s_C \equiv \bar{C}/\bar{Y}$. We have used the Taylor expansion

$$Y_t/\bar{Y} = 1 + \hat{Y}_t + \frac{1}{2}\hat{Y}_t^2 + \mathcal{O}(||\xi||^3)$$

to get a relation for $\tilde{Y}_t$ in terms of $\hat{Y}_t$. Finally the term "t.i.p." denotes terms that are independent of policy, and may accordingly be suppressed as far as the welfare ranking of alternative policies is concerned.

We may similarly approximate $v(Y_t; \xi_t)\Delta_t$ by

$$\begin{aligned} v(Y_t;\xi_t)\Delta_t &= \bar{v} + \bar{v}(\Delta_t - 1) + \bar{v}_y(Y_t - \bar{Y}) + \bar{v}_y(\Delta_t - 1)(Y_t - \bar{Y}) + (\Delta_t - 1)\bar{v}_\xi \xi_t + \\ &\quad + \frac{1}{2}\bar{v}_{yy}(Y_t - \bar{Y})^2 + (Y_t - \bar{Y})\bar{v}_{y\xi}\xi_t + \frac{1}{2}\xi_t' \bar{v}_{\xi\xi}\xi_t + \mathcal{O}(||\xi||^3) \\ &= \bar{v}(\Delta_t - 1) + \bar{v}_y \bar{Y}\left(\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2\right) + \bar{v}_y(\Delta_t - 1)\bar{Y}\hat{Y}_t + (\Delta_t - 1)\bar{v}_\xi \xi_t + \\ &\quad + \frac{1}{2}\bar{v}_{yy}\bar{Y}^2\hat{Y}_t^2 + \bar{Y}\hat{Y}_t \bar{v}_{y\xi}\xi_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\ &= \bar{v}_y \bar{Y}[\frac{\Delta_t - 1}{1+\omega} + \hat{Y}_t + \frac{1}{2}(1+\omega)\hat{Y}_t^2 + (\Delta_t - 1)\hat{Y}_t - \omega \hat{Y}_t q_t + \\ &\quad - \frac{\Delta_t - 1}{1+\omega}\omega q_t] + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \end{aligned}$$

We take a second-order expansion of (1.20), obtaining

$$\hat{\Delta}_t = \alpha \hat{\Delta}_{t-1} + \frac{\alpha}{1-\alpha}\theta(1+\omega)(1+\omega\theta)\frac{\pi_t^2}{2} + \mathcal{O}(||\xi||^3). \tag{A.43}$$

This in turn allows us to approximate $v(Y_t; \xi_t)\Delta_t$as

$$v(Y_t;\xi_t)\Delta_t = (1-\Phi)\bar{Y}u_c\left\{\frac{\hat{\Delta}_t}{1+\omega} + \hat{Y}_t + \frac{1}{2}(1+\omega)\hat{Y}_t^2 - \omega \hat{Y}_t q_t\right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \tag{A.44}$$

where we have used the steady state relation $\bar{v}_y = (1-\Phi)\bar{u}_c$to replace $\bar{v}_y$ by $(1-\Phi)\bar{u}_c$, and where

$$\Phi \equiv 1 - \left(\frac{\theta - 1}{\theta}\right)\left(\frac{1-\bar{\tau}}{\bar{\mu}^w}\right) < 1$$

measures the inefficiency of steady-state output $\bar{Y}$.

Combining (A.42) and (A.44), we finally obtain equation (2.1) in the text,

$$U_{t_0} = \bar{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Phi \hat{Y}_t - \frac{1}{2} u_{yy} \hat{Y}_t^2 + \hat{Y}_t u_{y\xi} \xi_t - u_\Delta \hat{\Delta}_t$$
$$+ \quad \text{t.i.p.} + \mathcal{O}(||\xi||^3), \qquad (A.45)$$

where

$$u_{yy} \equiv (\omega + \sigma^{-1}) - \Phi(1 + \omega),$$
$$u_{y\xi}\xi_t \equiv [\sigma^{-1} g_t + (1 - \Phi)\omega q_t],$$
$$u_\Delta \equiv \frac{(1 - \Phi)}{1 + \omega}.$$

We finally observe that (A.43) can be integrated to obtain

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0}\ \hat{\Delta}_t = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)} \theta(1+\omega)(1+\omega\theta) \sum_{t=t_0}^{\infty} \beta^{t-t_0}\ \frac{\pi_t^2}{2} + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \qquad (A.46)$$

By substituting (A.46) into (A.45), we obtain

$$U_{t_0} = \bar{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [\Phi \hat{Y}_t - \frac{1}{2} u_{yy} \hat{Y}_t^2 + \hat{Y}_t u_{y\xi} \xi_t - \frac{1}{2} u_\pi \pi_t^2]$$
$$+ \text{t.i.p.} + \mathcal{O}(||\xi||^3).$$

This coincides with equation (2.3) in the text, where we have further defined

$$\kappa \equiv \frac{(1-\alpha\beta)(1-\alpha)}{\alpha} \frac{(\omega + \sigma^{-1})}{(1+\theta\omega)}, \qquad u_\pi \equiv \frac{\theta(\omega + \sigma^{-1})(1 - \Phi)}{\kappa}.$$

A.3 A second-order approximation to the AS equation (equation (1.19))

We can write (1.13)

$$\tilde{p}_t = \left(\frac{K_t}{F_t}\right)^{\frac{1}{1+\omega\theta}},$$

where we have defined $\tilde{p}_t \equiv p_t^*/P_t$. We futher re-define the variables F_t and K_t as

$$F_t \equiv E_t \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} f_{t,T} \right\},$$
$$K_t \equiv E_t \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} k_{t,T} \right\},$$

with

$$f_{t,T} \equiv (1-\tau_T) f(Y_T;\xi_T) P_{t,T}^{1-\theta} = (1-\tau_T)\tilde{u}_c(C_T;\xi_T) Y_T P_{t,T}^{1-\theta}, \quad (A.47)$$

$$k_{t,T} \equiv k(Y_T;\xi_T) P_{t,T}^{-\theta(1+\omega)} = \frac{\theta}{\theta-1}\mu_T^w v_y(Y_T;\xi_T) Y_T P_{t,T}^{-\theta(1+\omega)}, \quad (A.48)$$

where we have defined $P_{t,T} \equiv P_t/P_T$. We can then obtain in an exact log-linear form that

$$(1+\omega\theta)\hat{\tilde{p}}_t + \hat{F}_t = \hat{K}_t. \quad (A.49)$$

We take a second-order expansion of F_t and K_t, obtaining

$$\hat{F}_t + \frac{1}{2}\hat{F}_t^2 = (1-\alpha\beta)E_t\left\{\sum_{T=t}^{+\infty}(\alpha\beta)^{T-t}(\hat{f}_{t,T} + \frac{1}{2}\hat{f}_{t,T}^2)\right\} + \mathcal{O}(||\xi||^3), \quad (A.50)$$

$$\hat{K}_t + \frac{1}{2}\hat{K}_t^2 = (1-\alpha\beta)E_t\left\{\sum_{T=t}^{+\infty}(\alpha\beta)^{T-t}(\hat{k}_{t,T} + \frac{1}{2}\hat{k}_{t,T}^2)\right\} + \mathcal{O}(||\xi||^3). \quad (A.51)$$

Plugging (A.50) and (A.51) into (A.49), we obtain

$$\begin{aligned}(1+\omega\theta)\hat{\tilde{p}}_t &= (1-\alpha\beta)E_t\left\{\sum_{T=t}^{+\infty}(\alpha\beta)^{T-t}(\hat{k}_{t,T} - \hat{f}_{t,T})\right\} + \\ &+ \frac{(1-\alpha\beta)}{2}E_t\left\{\sum_{T=t}^{+\infty}(\alpha\beta)^{T-t}(\hat{k}_{t,T}^2 - \hat{f}_{t,T}^2)\right\} + \\ &\frac{1}{2}(\hat{F}_t - \hat{K}_t)(\hat{F}_t + \hat{K}_t) + \mathcal{O}(||\xi||^3). \end{aligned} \quad (A.52)$$

We note that in an exact log-linear form

$$\hat{k}_{t,T} - \hat{f}_{t,T} = -(1+\omega\theta)\hat{P}_{t,T} + \omega(\hat{Y}_T - q_T) + \tilde{\sigma}^{-1}(\hat{C}_T - \bar{c}_T) - \hat{S}_T + \hat{\mu}_T^w,$$

where $\hat{S}_t = \ln(1-\tau_t)/(1-\bar{\tau})$.

Furthermore we obtain that

$$\begin{aligned}\hat{k}_{t,T} + \hat{f}_{t,T} &= (2+\omega)\hat{Y}_T - \omega q_T + (1-2\theta-\omega\theta)\hat{P}_{t,T} + \hat{\mu}_T^w + \hat{S}_T - \tilde{\sigma}^{-1}(\hat{C}_T - \bar{c}_T) \\ &= X_T + (1-2\theta-\omega\theta)\hat{P}_{t,T},\end{aligned}$$

where we have defined

$$X_T \equiv (2+\omega)\hat{Y}_T - \omega q_T + \hat{\mu}^w_T + \hat{S}_T - \tilde{\sigma}^{-1}(\hat{C}_T - \bar{c}_T).$$

We can then substitute into (A.52) and get

$$\begin{aligned}\hat{\tilde{p}}_t &= \frac{(1-\alpha\beta)}{(1+\omega\theta)} E_t \sum_{T=t}^{+\infty} (\alpha\beta)^{T-t}(\hat{k}_{t,T} - \hat{f}_{t,T}) + \\ &+ \frac{(1-\alpha\beta)}{2(1+\omega\theta)} E_t \sum_{T=t}^{+\infty} (\alpha\beta)^{T-t}[\omega(\hat{Y}_T - q_T) + \tilde{\sigma}^{-1}(\hat{C}_T - \bar{c}_T) - \hat{S}_T + \hat{\mu}^w_T][X_T + (1-2\theta-\omega\theta)\hat{P}_{t,T}] + \\ &- \frac{(1-\alpha\beta)}{2(1+\omega\theta)} E_t \sum_{T=t}^{+\infty} (\alpha\beta)^{T-t}(1+\omega\theta)[X_T + (1-2\theta-\omega\theta)\hat{P}_{t,T}]\hat{P}_{t,T} \\ &- \frac{(1-\alpha\beta)}{2} \hat{\tilde{p}}_t E_t \sum_{T=t}^{+\infty} (\alpha\beta)^{T-t}(X_T + (1-2\theta-\omega\theta)\hat{P}_{t,T}) + \mathcal{O}(||\xi||^3),\end{aligned}$$

which can be simplified to

$$\begin{aligned}\frac{(1+\omega\theta)}{(1-\alpha\beta)}\hat{\tilde{p}}_t &= -\frac{1}{2}(1+\omega\theta)\hat{\tilde{p}}_t Z_t + E_t \sum_{T=t}^{+\infty} (\alpha\beta)^{T-t}(\hat{k}_{t,T} - \hat{f}_{t,T}) + \\ &+ \frac{1}{2} E_t \sum_{T=t}^{+\infty} (\alpha\beta)^{T-t}[(\hat{k}_{t,T} - \hat{f}_{t,T})][X_T + (1-2\theta-\omega\theta)\hat{P}_{t,T}] + \\ &+ \mathcal{O}(||\xi||^3). \qquad \text{(A.53)}\end{aligned}$$

where we have defined

$$Z_t \equiv E_t \sum_{T=t}^{+\infty} (\alpha\beta)^{T-t}[X_T + (1-2\theta-\omega\theta)\hat{P}_{t,T}].$$

By using (A.53), and defining

$$z_T \equiv \hat{k}_{t,T} - \hat{f}_{t,T} + (1+\omega\theta)\hat{P}_{t,T},$$

we can write

$$\begin{aligned}\frac{(1+\omega\theta)}{(1-\alpha\beta)}\hat{\tilde{p}}_t &= z_t + \alpha\beta\frac{(1+\omega\theta)}{(1-\alpha\beta)} E_t(\hat{\tilde{p}}_{t+1} - \hat{P}_{t,t+1}) - \frac{1}{2}(1+\omega\theta)\hat{\tilde{p}}_t Z_t + \frac{1}{2}\alpha\beta(1+\omega\theta)E_t\hat{\tilde{p}}_{t+1}Z_{t+1} + \\ &+ \frac{1}{2} z_t X_t + \frac{\alpha\beta}{2} E_t\{\sum_{T=t+1}^{+\infty} (\alpha\beta)^{T-t-1}(1-2\theta-\omega\theta)(1+\omega\theta)(-\hat{P}^2_{t,t+1} - 2\hat{P}_{t,t+1}\hat{P}_{t+1,T}) + \\ &- (1+\omega\theta)\hat{P}_{t,t+1}X_T + (1-2\theta-\omega\theta)\hat{P}_{t,t+1}z_T\} + \mathcal{O}(||\xi||^3),\end{aligned}$$

which can be simplified to

$$\begin{aligned}
\frac{(1+\omega\theta)}{(1-\alpha\beta)}\hat{\tilde{p}}_t &= z_t + \alpha\beta\frac{(1+\omega\theta)}{(1-\alpha\beta)}E_t(\hat{\tilde{p}}_{t+1} - \hat{P}_{t,t+1}) + \frac{1}{2}z_t X_t + \\
&\quad -\frac{1}{2}(1+\omega\theta)\hat{\tilde{p}}_t Z_t + \frac{1}{2}\alpha\beta(1+\omega\theta)E_t\{(\hat{\tilde{p}}_{t+1} - \hat{P}_{t,t+1})Z_{t+1}\} \\
&\quad +\frac{\alpha\beta}{2(1-\alpha\beta)}(1-2\theta-\omega\theta)(1+\omega\theta)E_t\{(\hat{\tilde{p}}_{t+1} - \hat{P}_{t,t+1})\hat{P}_{t,t+1}\} + \\
&\quad +\mathcal{O}(||\xi||^3), \qquad \text{(A.54)}
\end{aligned}$$

We next take a second-order expansion of the law of motion (1.18) for the price index, obtaining

$$\hat{\tilde{p}}_t = \frac{\alpha}{1-\alpha}\pi_t - \frac{1-\theta}{2}\frac{\alpha}{(1-\alpha)^2}\pi_t^2 + \mathcal{O}(||\xi||^3), \qquad \text{(A.55)}$$

where we have used the fact that

$$\hat{\tilde{p}}_t = \frac{\alpha}{1-\alpha}\pi_t + \mathcal{O}(||\xi||^2),$$

and $\hat{P}_{t-1,t} = -\pi_t$. We can then plug (A.55) into (A.54) obtaining

$$\begin{aligned}
\pi_t &= \frac{1-\theta}{2}\frac{1}{(1-\alpha)}\pi_t^2 + \frac{\kappa}{(\omega+\sigma^{-1})}z_t + \beta E_t\pi_{t+1} - \frac{1-\theta}{2}\frac{\alpha\beta}{(1-\alpha)}E_t\pi_{t+1}^2 \\
&\quad +\frac{1}{2}\frac{\kappa}{(\omega+\sigma^{-1})}z_t X_t - \frac{1}{2}(1-\alpha\beta)\pi_t Z_t + \frac{\beta}{2}(1-\alpha\beta)E_t\{\pi_{t+1}Z_{t+1}\} \\
&\quad -\frac{\beta}{2}(1-2\theta-\omega\theta)E_t\{\pi_{t+1}^2\} + \mathcal{O}(||\xi||^3). \qquad \text{(A.56)}
\end{aligned}$$

By integrating equation (A.56) forward from time t_0 we can finally obtain

$$\begin{aligned}
V_{t_0} &= \frac{\kappa}{(\omega+\sigma^{-1})}E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}z_t + \frac{1}{2}\frac{\kappa}{(\omega+\sigma^{-1})}E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}z_t X_t \\
&\quad +\frac{\theta(1+\omega)}{2}E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\pi_t^2 + \mathcal{O}(||\xi||^3), \qquad \text{(A.57)}
\end{aligned}$$

where

$$V_{t_0} \equiv \pi_{t_0} - \frac{1-\theta}{2(1-\alpha)}\pi_{t_0}^2 + \frac{(1-\alpha\beta)}{2}\pi_{t_0}Z_{t_0} + \frac{\theta(1+\omega)}{2}\pi_{t_0}^2$$

and

$$Z_t = X_t - \frac{\alpha\beta}{1-\alpha\beta}(1-2\theta-\omega\theta)E_t\pi_{t+1} + \alpha\beta E_t Z_{t+1}.$$

Recalling that

$$z_t = \omega(\hat{Y}_t - q_t) + \tilde{\sigma}^{-1}(\hat{C}_t - \bar{c}_t) - \hat{S}_T + \hat{\mu}_t^w$$

we can take a second-order approximation of the relation between output and consumption $Y_t = C_t + G_t$ obtaining

$$\hat{C}_t = s_C^{-1}\hat{Y}_t - s_C^{-1}\hat{G}_t + \frac{s_C^{-1}(1 - s_C^{-1})}{2}\hat{Y}_t^2 + s_C^{-2}\hat{Y}_t\hat{G}_t + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3). \quad \text{(A.58)}$$

Finally

$$\hat{S}_t = -\omega_\tau \hat{\tau}_t + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3), \quad \text{(A.59)}$$

where $\omega_\tau \equiv \bar{\tau}/(1-\bar{\tau})$. Here "s.o.t.i.p" refers to second-order (or higher) terms independent of policy; the first-order terms have been kept as these will matter for the log-linear aggregate-supply relation that appears as a constraint in our policy problem. By substituting (A.58) and (A.59) into the definition of z_t in (A.56), we finally obtain a quadratic approximation to the AS relation.

$$\begin{aligned} V_{t_0} \;=\; & \frac{\kappa}{(\omega+\sigma^{-1})} E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}[(\omega+\sigma^{-1})\hat{Y}_t - \sigma^{-1}g_t - \omega q_t + \hat{\mu}_t^w + \omega_\tau\hat{\tau}_t] \\ & + \frac{1}{2}\frac{\kappa}{(\omega+\sigma^{-1})} E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}[(\omega+\sigma^{-1})(2+\omega-\sigma^{-1}) + \sigma^{-1}(1-s_C^{-1})]\hat{Y}_t^2 \\ & + \frac{\kappa}{(\omega+\sigma^{-1})} E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}[(1-\sigma^{-1})\omega_\tau\hat{\tau}_t + \sigma^{-1}s_C^{-1}\hat{G}_t - (1-\sigma^{-1})\sigma^{-1}g_t + \\ & -(1+\omega)\omega q_t + (1+\omega)\hat{\mu}_t^w]\hat{Y}_t + \\ & + \frac{\theta(1+\omega)}{2} E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\pi_t^2 + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \quad \text{(A.60)}$$

This can be expressed compactly in the form

$$\begin{aligned} V_{t_0} \;=\; & E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\kappa(\hat{Y}_t + c_\xi\xi_t + \frac{1}{2}c_{yy}\hat{Y}_t^2 - \hat{Y}_t c_{y\xi}\xi_t + \frac{1}{2}c_\pi\pi_t^2) \\ & + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned} \quad \text{(A.61)}$$

or as

$$\begin{aligned} V_t \;=\; & \kappa(\hat{Y}_t + c_\xi\xi_t + \frac{1}{2}c_{yy}\hat{Y}_t^2 - \hat{Y}_t c_{y\xi}\xi_t + \frac{1}{2}c_\pi\pi_t^2) + \beta E_t V_{t+1} \\ & + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned} \quad \text{(A.62)}$$

where we have defined

$$c_\xi \xi_t \equiv (\omega + \sigma^{-1})^{-1}[-\sigma^{-1} g_t - \omega q_t + \hat{\mu}_t^w + \omega_\tau \hat{\tau}_t].$$

$$c_{yy} \equiv (2 + \omega - \sigma^{-1}) + \sigma^{-1}(1 - s_C^{-1})(\omega + \sigma^{-1})^{-1}$$

$$\begin{aligned} c_{y\xi}\xi_t \quad &\equiv \quad (\omega + \sigma^{-1})^{-1}[-\sigma^{-1} s_C^{-1} \hat{G}_t + \sigma^{-1}(1 - \sigma^{-1}) g_t + \omega(1 + \omega) q_t \\ &\quad -(1 + \omega)\hat{\mu}_t^w - (1 - \sigma^{-1})\omega_\tau \hat{\tau}_t] \end{aligned}$$

$$c_\pi \equiv \frac{\theta(1 + \omega)}{\kappa}$$

and

$$V_t = \pi_t + \frac{1}{2} v_\pi \pi_t^2 + v_z \pi_t Z_t,$$

$$Z_t = z_y \hat{Y}_t + z_\pi \pi_t + z_\xi \xi_t + \alpha\beta E_t Z_{t+1},$$

in which the coefficients are defined as

$$v_\pi \equiv \theta(1 + \omega) - \frac{1 - \theta}{(1 - \alpha)}, \qquad v_z \equiv \frac{(1 - \alpha\beta)}{2},$$

$$v_k \equiv \frac{\kappa}{(\omega + \sigma^{-1})} \frac{\alpha}{1 - \alpha\beta}(1 - 2\theta - \omega\theta),$$

$$z_y \equiv (2 + \omega - \sigma^{-1}) + v_k(\omega + \sigma^{-1})$$

$$z_\xi \xi_t \equiv \sigma^{-1}(1 - v_k) g_t - \omega(1 + v_k) q_t + (1 + v_k)\hat{\mu}_t^w - \omega_\tau(1 - v_k)\hat{\tau}_t,$$

$$z_\pi \equiv -\frac{(\omega + \sigma^{-1})}{\kappa} v_k.$$

Note that in a first-order approximation, (A.62) can be written as simply

$$\pi_t = \kappa[\hat{Y}_t + c_\xi \xi_t] + \beta E_t \pi_{t+1}. \tag{A.63}$$

We can also write (A.61) as

$$\begin{aligned} V_{t_0} \quad &= \quad E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \kappa(\hat{Y}_t + \frac{1}{2} c_{yy} \hat{Y}_t^2 - \hat{Y}_t c_{y\xi} \xi_t + \frac{1}{2} c_\pi \pi_t^2) \\ &\quad + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{A.64}$$

where the term $c_\xi \xi_t$ is now included in terms independent of policy. (Such terms matter when part of the log-linear constraints, as in the case of (A.63), but not when part of the quadratic objective.)

A.4 Derivation of equation (2.7)

We can multiply equation (2.6) by $\Phi \bar{Y} \bar{u}_c$ and subtract from (2.1) to obtain

$$U_{t_0} = -\bar{Y}\bar{u}_c E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} q_y \hat{Y}_t^2 - \hat{Y}_t(u_{y\xi}\xi_t + \Phi c_{y\xi}\xi_t) + \frac{1}{2} q_\pi \pi_t^2 \right\} + T_{t_0} + \text{t.i.p.} + \mathcal{O}(||\xi||^3),$$

where

$$\begin{aligned} q_\pi &\equiv u_\pi + \Phi c_\pi \\ &= \frac{\theta(\omega + \sigma^{-1})(1-\Phi)}{\kappa} + \Phi \frac{\theta(1+\omega)}{\kappa} \\ &= \frac{\theta}{\kappa}[(\omega + \sigma^{-1}) + \Phi(1-\sigma^{-1})], \end{aligned}$$

$$\begin{aligned} q_y &\equiv u_{yy} + \Phi c_{yy} \\ &= (\omega + \sigma^{-1}) - \Phi(1+\omega) + \Phi(2 + \omega - \sigma^{-1}) + \Phi\sigma^{-1}(1 - s_C^{-1})(\omega + \sigma^{-1})^{-1} \\ &= (\omega + \sigma^{-1}) + \Phi(1 - \sigma^{-1}) + \frac{\Phi\sigma^{-1}(1 - s_C^{-1})}{\omega + \sigma^{-1}}. \end{aligned}$$

This can be rewritten in the form (2.7) given in the text, where

$$\begin{aligned} \hat{Y}_t^* &\equiv q_y^{-1}[u_{y\xi}\xi_t + \Phi c_{y\xi}\xi_t] \\ &= q_y^{-1}\{\sigma^{-1}g_t + (1-\Phi)\omega q_t + (\omega + \sigma^{-1})^{-1}\Phi[-\sigma^{-1}s_C^{-1}\hat{G}_t + \sigma^{-1}(1-\sigma^{-1})g_t + \omega(1+\omega)q_t \\ &\quad -(1+\omega)\hat{\mu}_t^w - (1-\sigma^{-1})\omega_\tau \hat{\tau}_t]\} \\ &= \omega_1 \hat{Y}_t^n - \omega_2 \hat{G}_t + \omega_3 \hat{u}_t^w + \omega_4 \hat{\tau}_t, \end{aligned}$$

and Ω, $\hat{Y}_t^n$, and the ω_i are defined as in the text.

A.5 General case: the deterministic steady state

We consider a stationary perfect foresight equilibrium for the recursive problem, in which all the shocks are constant $\xi_t = \bar{\xi}$. We wish to find initial values of the predetermined variables $D_{t_0-1} = \bar{D}$ and initial commitments $F_{t_0} = \bar{F}$ such that the optimal plan involves constant policy $\Pi_t = \bar{\Pi}$, $D_t = \bar{D}$, $\tilde{p}_t = \bar{p}$, $F_t = \bar{F}$, $Y_t = \bar{Y}$ for each time, where $\bar{D}$ and $\bar{F}$ are consistent with the initial values. Since $\tilde{p}_t = X_{0,t}$, we can write this program as maximizing

$$\sum_{t=t_0}^{\infty} \beta^{t-t_0} U(Y_t, D_t, \bar{\xi}) \tag{A.65}$$

by choosing $\{D_t,\, Y_t,\, F_{t+1},\, \Pi_t\}$ for each $t \geq t_0$ under the constraints

$$\gamma_0 h(D_{0,t}) + \sum_{j=1}^{\infty} \gamma_j h(D_{j-1,t-1}/\Pi_t) = 1 \tag{A.66}$$

$$D_{j,t} = D_{j-1,t-1}/\Pi_t \qquad \text{for each } j \geq 1 \tag{A.67}$$

$$\gamma_0 g(D_{0,t}; Y_t, \bar{\xi}) + \beta F_{1,t+1} = 0, \tag{A.68}$$

$$\gamma_j g(D_{j,t}; Y_t, \bar{\xi}) + \beta F_{j+1,t+1} = F_{j,t} \qquad \text{for each } j \geq 1 \tag{A.69}$$

and the initial conditions on the vector $F_{t_0} = \bar{F}$ and $D_{t_0-1} = \bar{D}$. We consider the Lagrangian problem. The first-order condition with respect to Y_t yields to

$$U_y(Y_t, D_t, \bar{\xi}) + \sum_{j=0}^{\infty} \varphi_{j,t}\gamma_j g_y(D_{j,t}; Y_t, \bar{\xi}) = 0 \tag{A.70}$$

where $\varphi_{j,t}$ are the Lagrangian multiplier associated with the constraints (A.68) and (A.69) for each $j \geq 1$. The first-order condition with respect to $D_{0,t}$ are

$$\begin{aligned} 0 \;=\; & U_{d_0}(Y_t, D_t, \bar{\xi}) + \phi_t \gamma_0 h_{d_0}(D_{0,t}) + \beta\phi_{t+1}\gamma_1 h_{d_1}(D_{0,t}/\Pi_{t+1})\Pi_{t+1}^{-1} \\ & -\beta\eta_{1,t+1}\Pi_{t+1}^{-1} + \varphi_{0,t}\gamma_0 g_{d_0}(D_{0,t}; Y_t, \bar{\xi}) \end{aligned} \tag{A.71}$$

where ϕ_t is the Lagrangian multiplier associated with the constraint (A.66) while $\eta_{j,t}$ with $j \geq 1$ are the multipliers associated with the constraints (A.67). The first-order conditions with respect to $D_{j,t}$ are for each $j \geq 1$

$$\begin{aligned} 0 \;=\; & U_{d_j}(Y_t, X_t, \bar{\xi}) + \beta\phi_{t+1}\gamma_{j+1} h_{d_{j+1}}(D_{j,t}/\Pi_{t+1})\Pi_{t+1}^{-1} + \eta_{j,t} - \beta\eta_{j+1,t+1}\Pi_{t+1}^{-1} \\ & +\varphi_{j,t}\gamma_j g_{d_j}(D_{j,t}; Y_t, \bar{\xi}). \end{aligned} \tag{A.72}$$

The first-order condition with respect to Π_t yields

$$\phi_t \sum_{j=1}^{\infty} \gamma_j h_{d_j}\left(\frac{D_{j-1,t-1}}{\Pi_t}\right)\left(\frac{D_{j-1,t-1}}{\Pi_t^2}\right) = \sum_{j=1}^{\infty} \eta_{j,t}\left(\frac{D_{j-1,t-1}}{\Pi_t^2}\right) \tag{A.73}$$

The first-order condition with respect to $F_{j,t}$ are

$$\varphi_{j,t-1} = \varphi_{j+1,t} \tag{A.74}$$

for each $j \geq 0$ and each $t \geq t_0$. In deriving (A.74), we have appropriately standardized the multiplier to the initial vector of constraints $F_t = \bar{F}$ in a way that the first-order conditions were time invariant. We are looking for a steady-state solution to this optimization problem. In this case $\varphi_j = \bar{\varphi}$ for all $j \geq 0$. Moreover we note that

$$U_{d_j}(Y_t, D_t; \bar{\xi}) = -\gamma_j v_{d_j}(Y_t, D_{j,t}; \bar{\xi}).$$

In particular we look for a steady state in which $\bar{\Pi} = 1$ and consequently the vector of $\bar{D}$ is all composed by ones. In particular we observe that in this steady state, $v_{d_j}(\bar{Y}, 1; \bar{\xi}) = v_d(\bar{Y}, 1; \bar{\xi})$, $g_{d_j}(\bar{Y}, 1, \bar{\xi}) = g_d(\bar{Y}, 1, \bar{\xi})$,and $h_{d_j}(1) = h_d(1)$ for all $j \geq 0$. Moreover in this steady state $g(\bar{Y}, 1, \bar{\xi}) = 0$ which implies

$$1 = \frac{\bar{\mu}^w}{1-\bar{\tau}} \frac{\bar{\theta}}{(1-\bar{\theta})} \frac{v_y(\bar{Y}, \bar{\xi})}{u_y(\bar{Y}; \bar{\xi})}$$

which defines the steady-state of output where $\bar{\theta} = \theta(1)$. Moreover $\bar{F} = 0$. Furthermore we note that (A.70) implies

$$U_y(\bar{Y}, 1, \bar{\xi}) + \bar{\varphi} g_y(1; \bar{Y}, \bar{\xi}) = 0 \tag{A.75}$$

while (A.71), (A.72) implies

$$0 = -\gamma_0 v_d(\bar{Y}, 1; \bar{\xi}) + \bar{\phi}\gamma_0 h_d(1) + \beta\bar{\phi}\gamma_1 h_d(1) - \beta\bar{\eta}_1 + \bar{\varphi}\gamma_0 g_d(\bar{Y}, 1, \bar{\xi}) \tag{A.76}$$

$$0 = -\gamma_j v_d(\bar{Y}, 1; \bar{\xi}) + \beta\bar{\phi}\gamma_{j+1} h_d(1) + \bar{\eta}_j - \beta\bar{\eta}_{j+1} + \bar{\varphi}\gamma_j g_d(\bar{Y}, 1, \bar{\xi}), \tag{A.77}$$

for each $j \geq 1$ while (A.73) implies

$$\bar{\phi} h_d(1)(1-\gamma_0) = \sum_{j=1}^{\infty} \bar{\eta}_j \tag{A.78}$$

We can sum (A.76)-(A.77) to obtain

$$-v_d(\bar{Y}, 1; \bar{\xi}) + \bar{\phi}\gamma_0 h_d(1) + \beta\bar{\phi}(1-\gamma_0) h_d(1) + (1-\beta)\sum_{j=1}^{\infty} \bar{\eta}_j + \bar{\varphi} g_d(\bar{Y}, 1, \bar{\xi}) = 0$$

in which we can substitute (A.78) to obtain

$$-v_d(\bar{Y}, 1; \bar{\xi}) + \bar{\phi} h_d(1) + \bar{\varphi} g_d(\bar{Y}, 1, \bar{\xi}) = 0. \tag{A.79}$$

Thus equations (A.75) and (A.79) determines the Lagrangian multipliers $\bar{\phi}$ and $\bar{\varphi}$ while the set of equations (A.76) and (A.77) determine the multipliers $\bar{\eta}_j$ for each $j \geq 1$. In particular, $\bar{\eta}_j = \bar{\phi}\gamma_j h_d(1)$ for each j. We have then found that if there is no initial price dispersions, and if policymakers are restricted to take further commitments on promised value of variables, then stability of the price level is a candidate for an optimal steady-state policy. This result is robust to the general specification of price-setting mechanisms chosen.

A.6 General case : a second-order approximation to utility (equation (4.13))

We first consider the utility flow

$$U_t = u(Y_t;\xi_t) - \sum_{j=0}^{\infty}\gamma_j v(Y_t z(D_{j,t});\xi_t). \tag{A.80}$$

We can expand $u(Y_t;\xi_t)$ as it follows

$$\begin{aligned} u(Y_t;\xi_t) &= \bar{u} + \bar{u}_y(Y_t-\bar{Y}) + \bar{u}_\xi\xi_t + \frac{1}{2}\bar{u}_{yy}(Y_t-\bar{Y})^2 + \bar{u}_{y\xi}\xi_t(Y_t-\bar{Y}) + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3) \\ &= \bar{u}_y\bar{Y}[\hat{Y}_t + \frac{1}{2}(1-\sigma^{-1})\hat{Y}_t^2 + \sigma^{-1}g_t\hat{Y}_t] + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned} \tag{A.81}$$

where $\sigma^{-1} \equiv -\bar{u}_{yy}\bar{Y}/\bar{u}_y$ and $g_t \equiv \sigma\bar{u}_{y\xi}\xi_t/\bar{u}_y$. We now expand $v(Y_t z(D_{j,t});\xi_t)$ obtaining

$$\begin{aligned} v(Y_t z(D_{j,t});\xi_t) &= \bar{v} + \bar{v}_y\bar{z}(Y_t-\bar{Y}) + \bar{v}_y\bar{Y}\bar{z}'(D_{j,t}-1) + \bar{v}_\xi\xi_t + \\ &+\frac{1}{2}\bar{v}_{yy}\bar{z}^2(Y_t-\bar{Y})^2 + \frac{1}{2}(\bar{v}_{yy}(\bar{z}')^2\bar{Y}^2 + \bar{v}_y\bar{Y}\bar{z}'')(D_{j,t}-1)^2 + \\ &+(\bar{v}_{yy}\bar{z}\bar{Y}\bar{z}' + \bar{v}_y\bar{z}')(Y_t-\bar{Y})(D_{j,t}-1) + [\bar{z}(Y_t-\bar{Y}) + \bar{Y}\bar{z}'(D_{j,t}-1)]\bar{v}_{y\xi}\xi_t + \\ &+\text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned}$$

which can be simplified to

$$\begin{aligned} v(Y_t z(D_{j,t});\xi_t) &= \bar{v}_y\bar{Y}[\hat{Y}_t + \frac{1}{2}(1+\omega)\hat{Y}_t^2 + \bar{z}'\hat{D}_{j,t} + \bar{v}_\xi\xi_t + \frac{1}{2}(\omega(\bar{z}')^2 + \bar{z}' + \bar{z}'')\hat{D}_{j,t}^2 + \\ &+\bar{z}'(1+\omega)\hat{Y}_t\hat{D}_{j,t} - (\hat{Y}_t + \bar{z}'\hat{D}_{j,t})\omega q_t] + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned}$$

where we have used $\bar{z} = z(1) = 1$ and defined $\omega \equiv \bar{v}_{yy}\bar{Y}/\bar{v}_y$ and $q_t \equiv -\omega^{-1}\bar{v}_{y\xi}\xi_t/\bar{v}_y$.

We now consider the constraint

$$\sum_{j=0}^{\infty}\gamma_j h(D_{j,t}) = 1$$

and take a second-order approximation of it obtaining

$$E_j\hat{D}_{j,t} + \frac{1}{2}(\bar{h}')^{-1}(\bar{h}' + \bar{h}'')E_j\hat{D}_{j,t}^2 + \mathcal{O}(||\xi||^3) = 0 \tag{A.82}$$

where we have defined

$$E_j\hat{D}_{j,t} \equiv \sum_{j=0}^{\infty}\gamma_j\hat{D}_{j,t} \qquad\qquad E_j\hat{D}_{j,t}^2 \equiv \sum_{j=0}^{\infty}\gamma_j\hat{D}_{j,t}^2.$$

We can now write

$$\sum_{j=0}^{\infty} \gamma_j v(Y_t z(D_{j,t}); \xi_t) = \bar{v}_y \bar{Y}[\hat{Y}_t + \frac{1}{2}(\omega(\bar{z}')^2 + \bar{z}' + \bar{z}'' - \bar{z}'(\bar{h}')^{-1}(\bar{h}' + \bar{h}''))E_j \hat{D}_{j,t}^2 + + \frac{1}{2}(1+\omega)\hat{Y}_t^2 - \omega \hat{Y}_t q_t] + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \tag{A.83}$$

We now note that in the steady state, $\bar{u}_c = (1 - \Phi)\bar{v}_y$. We can now substitute (A.81) and (A.83) into (A.80) to obtain

$$U_t = \bar{u}_y \bar{Y}\{\Phi \hat{Y}_t - \frac{1}{2}[(\omega + \sigma^{-1}) - \Phi(1+\omega)]\hat{Y}_t^2 + [\sigma^{-1} g_t + (1-\Phi)\omega q_t]\hat{Y}_t + -\frac{1}{2}(1-\Phi)[\omega(\bar{z}')^2 + \bar{z}' + \bar{z}'' - \bar{z}'(\bar{h}')^{-1}(\bar{h}' + \bar{h}'')]E_j \hat{D}_{j,t}^2\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3).$$

We can take the expected discounted sum across time and states of nature and obtain

$$U_{t_0} = \bar{u}_y \bar{Y} \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{\Phi \hat{Y}_t - \frac{1}{2} u_{yy} \hat{Y}_t^2 + \hat{Y}_t u_{y\xi} \xi_t - \frac{1}{2} u_d E_j \hat{D}_{j,t}^2\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \tag{A.84}$$

which is equation (4.13) in the text where we have defined

$$u_{yy} \equiv [(\omega + \sigma^{-1}) - \Phi(1+\omega)],$$

$$u_{y\xi}\xi_t \equiv [\sigma^{-1} g_t + (1-\Phi)\omega q_t],$$

and

$$u_d \equiv (1-\Phi)[\omega(\bar{z}')^2 + \bar{z}' + \bar{z}'' - \bar{z}'(\bar{h}')^{-1}(\bar{h}' + \bar{h}'')].$$

A.7 General case: a second-order approximation to the price constraints (equation (4.17))

We now consider a second-order approximation to the generic constraint

$$\gamma_j g(D_{j,t}, Y_t; \xi_t) + \beta F_{j+1,t+1} = F_{j,t} \tag{A.85}$$

where

$$g(D_{j,t}, Y_t; \xi_t) = u_y(Y_t; \xi_t) z(D_{j,t}) Y_t M_{j,t}$$

where

$$M_{j,t} \equiv (1-\tau_t) D_{j,t}(1 - \theta(D_{j,t})) + \mu_t^w \theta(D_{j,t}) \frac{v_y(Y_t z(D_{j,t}), \xi_t)}{u_y(Y_t; \xi_t)},$$

while $\theta(D_{j,t}) = -D_{j,t}z'(D_{j,t})/z(D_{j,t})$. We note that in the steady-state $\bar{M}_j = 0$, and as well $\bar{F}_j = 0$ for all j and $\bar{\theta} = \theta(1)$. We can take a second-order approximation to (A.85) obtaining

$$\begin{aligned} & \gamma_j \bar{u}_y \bar{z} \bar{Y} M_{j,t} + \gamma_j [\bar{u}_{yy} \bar{z} \bar{Y}(Y_t - \bar{Y}) + \bar{Y} \bar{z} \bar{u}_{y\xi} \xi_t + \bar{u}_y \bar{z}(Y_t - \bar{Y}) + \bar{u}_y \bar{Y} \bar{z}'(D_{j,t} - 1)] M_{j,t} \\ = \; & F_{j,t} - \beta F_{j+1,t+1} + \mathcal{O}(||\xi||^3). \end{aligned}$$

which can be rewritten as

$$\gamma_j M_{j,t} + \gamma_j [(1 - \sigma^{-1}) \hat{Y}_t + \sigma^{-1} g_t + \bar{z}' \hat{D}_{j,t}] M_{j,t} = \bar{u}_y^{-1} \bar{Y}^{-1} (F_{j,t} - \beta F_{j+1,t+1}) + \mathcal{O}(||\xi||^3). \tag{A.86}$$

since $\bar{z} = 1$ which is equation (4.15). We now take a second-order approximation to $M_{j,t}$ obtaining

$$\begin{aligned} M_{j,t} = \; & (1-\bar{\tau})(1-\bar{\theta})(D_{j,t}-1) - (1-\bar{\tau})\bar{\theta}'(D_{j,t}-1) - (1-\bar{\tau})\bar{\theta}'(D_{j,t}-1)^2 \\ & -\frac{1}{2}(1-\bar{\tau})\bar{\theta}''(D_{j,t}-1)^2 - (1-\bar{\theta})(\tau_t - \bar{\tau}) \\ & +\bar{\theta}\frac{\bar{v}_y}{\bar{u}_y}(\mu_t^w - \bar{\mu}) + \bar{\mu}\frac{\bar{v}_y}{\bar{u}_y}\bar{\theta}'(D_{j,t}-1) + \bar{\mu}\frac{1}{\bar{u}_y}\bar{\theta}[\bar{v}_{yy}\bar{z}(Y_t - \bar{Y}) + \bar{v}_{yy}\bar{Y}\bar{z}'(D_{j,t}-1) + \\ & +\bar{v}_{y\xi}\xi_t] - \bar{\theta}\frac{\bar{v}_y}{(\bar{u}_y)^2}\bar{\mu}[\bar{u}_{yy}(Y_t - \bar{Y}) + \bar{u}_{y\xi}\xi_t] + \bar{\theta}'\frac{\bar{v}_y}{\bar{u}_y}(\mu_t^w - \bar{\mu})(D_{j,t}-1) \\ & +\bar{\theta}\frac{1}{\bar{u}_y}(\mu_t^w - \bar{\mu})[\bar{v}_{yy}\bar{z}(Y_t - \bar{Y}) + \bar{v}_{yy}\bar{Y}\bar{z}'(D_{j,t}-1) + \bar{v}_{y\xi}\xi_t] + \\ & -\bar{\theta}\frac{\bar{v}_y}{[\bar{u}_y]^2}(\mu_t^w - \bar{\mu})[\bar{u}_{yy}(Y_t - \bar{Y}) + \bar{u}_{y\xi}\xi_t] + \frac{1}{2}\bar{\mu}\frac{\bar{v}_y}{\bar{u}_y}\bar{\theta}''(D_{j,t}-1)^2 \\ & +\bar{\mu}\frac{1}{\bar{u}_y}\bar{\theta}'(D_{j,t}-1)[\bar{v}_{yy}\bar{z}(Y_t - \bar{Y}) + \bar{v}_{yy}\bar{Y}\bar{z}'(D_{j,t}-1) + \bar{v}_{y\xi}\xi_t] + \\ & -\bar{\mu}\frac{\bar{v}_y}{[\bar{u}_y]^2}\bar{\theta}'(D_{j,t}-1)[\bar{u}_{yy}(Y_t - \bar{Y}) + \bar{u}_{y\xi}\xi_t] + \frac{1}{2}\bar{\mu}\frac{1}{\bar{u}_y}\bar{\theta}\bar{v}_{yyy}\bar{z}^2(Y_t - \bar{Y})^2 \\ & +\frac{1}{2}\bar{\mu}\frac{1}{\bar{u}_y}\bar{\theta}[\bar{v}_{yyy}\bar{Y}^2(\bar{z}')^2 + \bar{v}_{yy}\bar{Y}\bar{z}''](D_{j,t}-1)^2 + \bar{\mu}\frac{1}{\bar{u}_y}\bar{\theta}[\bar{v}_{yy}\bar{z}' + \bar{v}_{yyy}\bar{Y}\bar{z}\bar{z}'](D_{j,t}-1)(Y_t - \bar{Y}) \\ & +\bar{\mu}\frac{1}{\bar{u}_y}\bar{\theta}[\bar{z}(Y_t - \bar{Y})\bar{v}_{yy\xi}\xi_t + \bar{Y}\bar{z}'(D_{j,t}-1)\bar{v}_{yy\xi}\xi_t] - \bar{\mu}\frac{1}{[\bar{u}_y]^2}\bar{\theta}[\bar{v}_{yy}\bar{z}(Y_t - \bar{Y}) + \\ & +\bar{v}_{yy}\bar{Y}\bar{z}'(D_{j,t}-1) + \bar{v}_{y\xi}\xi_t] \cdot [\bar{u}_{yy}(Y_t - \bar{Y}) + \bar{u}_{y\xi}\xi_t] - \frac{1}{2}\bar{\theta}\frac{\bar{v}_y}{(\bar{u}_y)^4}\bar{\mu}[\bar{u}_{yyy}\bar{u}_y^2 + \\ & -2\bar{u}_{yy}^2\bar{u}_y](Y_t - \bar{Y})^2 - \bar{\theta}\frac{\bar{v}_y}{(\bar{u}_y)^4}\bar{\mu}[\bar{u}_y^2(Y_t - \bar{Y})\bar{u}_{yy\xi}\xi_t + \\ & -2\bar{u}_y\bar{u}_{yy}(Y_t - \bar{Y})\bar{u}_{y\xi}\xi_t] + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned}$$

which can be simplified to

$$\begin{aligned}
b_m M_{j,t} \;=\;& -(\hat{D}_{j,t}+\frac{1}{2}\hat{D}^2_{j,t})-(\bar{\theta}-1)^{-1}\bar{\theta}'(\hat{D}_{j,t}+\frac{1}{2}\hat{D}^2_{j,t})-(\bar{\theta}-1)^{-1}\bar{\theta}'\hat{D}^2_{j,t}+\omega_\tau\hat{\tau}_t \\
& -\frac{1}{2}(\bar{\theta}-1)^{-1}\bar{\theta}''\hat{D}^2_{j,t}+\hat{\mu}^w_t+\bar{\theta}^{-1}\bar{\theta}'(\hat{D}_{j,t}+\frac{1}{2}\hat{D}^2_{j,t})+[\frac{\bar{v}_{yy}\bar{Y}}{\bar{v}_y}(\hat{Y}_t+\frac{1}{2}\hat{Y}^2_t)+ \\
& +\frac{\bar{v}_{yy}\bar{Y}}{\bar{v}_y}\bar{z}'(\hat{D}_{j,t}+\frac{1}{2}\hat{D}^2_{j,t})+\frac{\bar{v}_{y\xi}}{\bar{v}_y}\xi_t]-[\frac{\bar{u}_{yy}\bar{Y}}{\bar{u}_y}(\hat{Y}_t+\frac{1}{2}\hat{Y}^2_t)+\frac{\bar{u}_{y\xi}}{\bar{u}_y}\xi_t]+ \\
& +\bar{\theta}'\bar{\theta}^{-1}\hat{\mu}^w_t\hat{D}_{j,t}+\hat{\mu}^w_t[\frac{\bar{v}_{yy}\bar{Y}}{\bar{v}_y}\hat{Y}_t+\frac{\bar{v}_{yy}\bar{Y}}{\bar{v}_y}\bar{z}'\hat{D}_{j,t}+\frac{\bar{v}_{y\xi}}{\bar{v}_y}\xi_t]+ \\
& -\hat{\mu}^w_t[\frac{\bar{u}_{yy}\bar{Y}}{\bar{u}_y}\hat{Y}_t+\frac{\bar{u}_{y\xi}}{\bar{u}_y}\xi_t]+\frac{1}{2}\bar{\theta}^{-1}\bar{\theta}''\hat{D}^2_{j,t}+\bar{\theta}'\bar{\theta}^{-1}\hat{D}_{j,t}[\frac{\bar{v}_{yy}\bar{Y}}{\bar{v}_y}\hat{Y}_t+\frac{\bar{v}_{yy}\bar{Y}}{\bar{v}_y}\bar{z}'\hat{D}_{j,t}+ \\
& +\frac{\bar{v}_{y\xi}}{\bar{v}_y}\xi_t]-\bar{\theta}^{-1}\bar{\theta}'\hat{D}_{j,t}[\frac{\bar{u}_{yy}\bar{Y}}{\bar{u}_y}\hat{Y}_t+\frac{\bar{u}_{y\xi}}{\bar{u}_y}\xi_t]+\frac{1}{2}\frac{\bar{v}_{yyy}\bar{Y}^2}{\bar{v}_y}\hat{Y}^2_t+ \\
& +\frac{1}{2}[\frac{\bar{v}_{yyy}\bar{Y}^2}{\bar{v}_y}(\bar{z}')^2+\frac{\bar{v}_{yy}\bar{Y}}{\bar{v}_y}\bar{z}'']\hat{D}^2_{j,t}+[\frac{\bar{v}_{yy}\bar{Y}\bar{z}'}{\bar{v}_y}+\frac{\bar{v}_{yyy}\bar{Y}^2\bar{z}'}{\bar{v}_y}]\hat{D}_{j,t}\hat{Y}_t \\
& +[\frac{\bar{v}_{yy\xi}\xi_t\bar{Y}}{\bar{v}_y}\hat{Y}_t+\frac{\bar{v}_{yy\xi}\xi_t\bar{Y}}{\bar{v}_y}\bar{z}'\hat{D}_{j,t}]-[\frac{\bar{v}_{yy}\bar{Y}}{\bar{v}_y}\hat{Y}_t+\frac{\bar{v}_{yy}\bar{Y}\bar{z}'}{\bar{v}_y}\hat{D}_{j,t}+\frac{\bar{v}_{y\xi}\xi_t}{\bar{v}_y}]\cdot \\
& [\frac{\bar{u}_{yy}\bar{Y}}{\bar{u}_y}\hat{Y}_t+\frac{\bar{u}_{y\xi}}{\bar{u}_y}\xi_t]-\frac{1}{2}[\frac{\bar{u}_{yyy}\bar{Y}^2}{\bar{u}_y}-2\frac{\bar{u}^2_{yy}\bar{Y}^2}{\bar{u}^2_y}]\hat{Y}^2_t \\
& -[\hat{Y}_t\frac{\bar{u}_{yy\xi}\bar{Y}}{\bar{u}_y}\xi_t-2\frac{\bar{u}_{yy}\bar{Y}}{\bar{u}^2_y}\hat{Y}_t\bar{u}_{y\xi}\xi_t]+\text{s.o.t.i.p.}+\mathcal{O}(||\xi||^3)
\end{aligned}$$

where we have used the fact that $\bar{z}=1$ and defined $b_m \equiv (1-\bar{\tau})^{-1}(\bar{\theta}-1)^{-1}$ and $\omega_\tau \equiv \frac{\bar{\tau}}{1-\bar{\tau}}$. We can now use the following definitions

$$\omega \equiv \frac{\bar{v}_{yy}\bar{Y}}{\bar{v}_y} \qquad \sigma^{-1} \equiv -\frac{\bar{u}_{yy}\bar{Y}}{\bar{u}_y}$$

$$g_t \equiv \sigma\frac{\bar{u}_{y\xi}}{\bar{u}_y}\xi_t \qquad q_t \equiv -\omega^{-1}\frac{\bar{v}_{y\xi}}{\bar{v}_y}\xi_t$$

$$\tilde{\omega}_1 \equiv \frac{\bar{v}_{yyy}\bar{Y}^2}{\bar{v}_y} \qquad \sigma_1^{-1} \equiv -\frac{\bar{u}_{yyy}\bar{Y}^2}{\bar{u}_y}$$

$$q_{2,t} \equiv -\tilde{\omega}_2^{-1}\frac{\bar{v}_{yy\xi}\xi_t\bar{Y}}{\bar{v}_y} \qquad g_{2,t} \equiv \sigma_2\frac{\bar{u}_{yy\xi}\bar{Y}}{\bar{u}_y}\xi_t$$

and simplify the above expression to

$$
\begin{aligned}
b_m M_{j,t} &= (-1-(\bar\theta-1)^{-1}\bar\theta'+\bar\theta^{-1}\bar\theta'+\omega\bar z')\hat D_{j,t}+\omega_\tau\hat\tau_t \\
&+\frac{1}{2}(-1-(\bar\theta-1)^{-1}\bar\theta'-2(\bar\theta-1)^{-1}\bar\theta'-(\bar\theta-1)^{-1}\bar\theta''+\bar\theta^{-1}\bar\theta'+\omega\bar z')\hat D_{j,t}^2 \\
&+\hat\mu_t^w+[\omega(\hat Y_t+\frac{1}{2}\hat Y_t^2)-\omega q_t]-[-\sigma^{-1}(\hat Y_t+\frac{1}{2}\hat Y_t^2)+\sigma^{-1}g_t]+ \\
&+\bar\theta'\bar\theta^{-1}\hat\mu_t^w\hat D_{j,t}+\hat\mu_t^w[\omega\hat Y_t+\omega\bar z'\hat D_{j,t}-\omega q_t]+ \\
&-\hat\mu_t^w[-\sigma^{-1}\hat Y_t+\sigma^{-1}g_t]+\frac{1}{2}\bar\theta^{-1}\bar\theta''\hat D_{j,t}^2+\bar\theta'\bar\theta^{-1}\hat D_{j,t}[\omega\hat Y_t+\omega\bar z'\hat D_{j,t}+ \\
&-\omega q_t]-\bar\theta^{-1}\bar\theta'\hat D_{j,t}[-\sigma^{-1}\hat Y_t+\sigma^{-1}g_t]+\frac{1}{2}\tilde\omega_1\hat Y_t^2+ \\
&+\frac{1}{2}[\tilde\omega_1(\bar z')^2+\omega\bar z'']\hat D_{j,t}^2+[\omega\bar z'+\tilde\omega_1\bar z']\hat D_{j,t}\hat Y_t+ \\
&-[\hat Y_t\tilde\omega_2 q_{2,t}+\bar z'\hat D_{j,t}\tilde\omega_2 q_{2,t}]-[\omega\hat Y_t+\omega\bar z'\hat D_{j,t}-\omega q_t]\cdot \\
&[-\sigma^{-1}\hat Y_t+\sigma^{-1}g_t]-\frac{1}{2}[-\sigma_1^{-1}-2\sigma^{-2}]\hat Y_t^2 \\
&-[\hat Y\sigma_2^{-1}g_{2,t}+2\sigma^{-2}\hat Y_t g_t]+\text{s.o.t.i.p.}+\mathcal{O}(||\xi||^3)
\end{aligned}
$$

and then

$$
\begin{aligned}
b_m M_{j,t} &= (-1-\bar\theta^{-1}(\bar\theta-1)^{-1}\bar\theta'+\omega\bar z')\hat D_{j,t}+\hat\mu_t^w+\omega_\tau\hat\tau_t-\omega q_t-\sigma^{-1}g_t+(\omega+\sigma^{-1})\hat Y_t \\
&+\frac{1}{2}(-1-(\bar\theta-1)^{-1}\bar\theta^{-1}\bar\theta'-2(\bar\theta-1)^{-1}\bar\theta'-(\bar\theta-1)^{-1}\bar\theta^{-1}\bar\theta''+ \\
&+2\bar\theta'\bar\theta^{-1}\omega\bar z'+\omega\bar z'+\tilde\omega_1(\bar z')^2+\omega\bar z'')\hat D_{j,t}^2 \\
&+\frac{1}{2}(\omega+\sigma^{-1}+\omega_1+\sigma_1^{-1}+2\sigma^{-2}+2\omega\sigma^{-1})\hat Y_t^2+ \\
&+(\bar\theta'\bar\theta^{-1}+\omega\bar z')\hat\mu_t^w\hat D_{j,t}+(\omega+\sigma^{-1})\hat\mu_t^w\hat Y_t+ \\
&-[\sigma_2^{-1}g_{2,t}+2\sigma^{-2}g_t+\tilde\omega_2 q_{2,t}+\omega\sigma^{-1}g_t+\omega\sigma^{-1}q_t]\hat Y_t+ \\
&+[(\omega+\sigma^{-1})\bar\theta'\bar\theta^{-1}+\omega\bar z'+\tilde\omega_1\bar z'+\omega\bar z'\sigma^{-1}]\hat D_{j,t}\hat Y_t-[\bar\theta^{-1}\bar\theta'(\omega q_t+\sigma^{-1}g_t)+ \\
&+\bar z'\tilde\omega_2 q_{2,t}+\omega\bar z'\sigma^{-1}g_t]\hat D_{j,t}+\text{s.o.t.i.p.}+\mathcal{O}(||\xi||^3) \qquad \text{(A.87)}
\end{aligned}
$$

We now sum (A.86) across j and obtain

$$
\bar u_c^{-1}\bar Y^{-1}(\tilde F_t-\beta\tilde F_{t+1})=E_jM_{j,t}+[(1-\sigma^{-1})\hat Y_t+\sigma^{-1}g_t]E_jM_{j,t}+\bar z'E_j\{\hat D_{j,t}M_{j,t}\}+\mathcal{O}(||\xi||^3). \qquad \text{(A.88)}
$$

where we have defined

$$
\tilde F_t \equiv \sum_{j=1}^{\infty} F_{j,t}.
$$

We now note that (A.82) and (A.87) imply

$$
\begin{aligned}
b_m E_j M_{j,t} \;=\;& \hat{\mu}_t^w + \omega_\tau \hat{\tau}_t - \omega q_t - \sigma^{-1} g_t + (\omega + \sigma^{-1})\hat{Y}_t + \frac{1}{2}(-1 - (\bar{\theta}-1)^{-1}\bar{\theta}^{-1}\bar{\theta}' - (\bar{\theta}-1)^{-1}\bar{\theta}^{-1}\bar{\theta}'' \\
& -2(\bar{\theta}-1)^{-1}\bar{\theta}' + 2\bar{\theta}'\bar{\theta}^{-1}\omega\bar{z}' + \omega\bar{z}' + \tilde{\omega}_1(\bar{z}')^2 + \omega\bar{z}'')E_j\hat{D}_{j,t}^2 - \\
& -\frac{1}{2}(\bar{h}')^{-1}(\bar{h}' + \bar{h}'')(-1 - \bar{\theta}^{-1}(\bar{\theta}-1)^{-1}\bar{\theta}' + \omega\bar{z}')E_j\hat{D}_{j,t}^2 + \\
& +\frac{1}{2}(\omega + \sigma^{-1} + \tilde{\omega}_1 + \sigma_1^{-1} + 2\sigma^{-2} + 2\omega\sigma^{-1})\hat{Y}_t^2 + (\omega + \sigma^{-1})\hat{\mu}_t^w\hat{Y}_t + \\
& -[\sigma_2^{-1}g_{2,t} + 2\sigma^{-2}g_t + \tilde{\omega}_2 q_{2,t} + \omega\sigma^{-1}g_t + \omega\sigma^{-1}q_t]\hat{Y}_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \qquad \text{(A.89)}
\end{aligned}
$$

and that

$$
(\bar{\theta}-1)^{-1}E_j\{\hat{D}_{j,t}M_{j,t}\} = (-1 - \bar{\theta}^{-1}(\bar{\theta}-1)^{-1}\bar{\theta}' + \omega\bar{z}')E_j\hat{D}_{j,t}^2 \qquad \text{(A.90)}
$$

We can then substitute (A.89) and (A.90) into (A.88) and obtain

$$
\begin{aligned}
b_m\bar{u}_c^{-1}\bar{Y}^{-1}(\tilde{F}_t - \beta\tilde{F}_{t+1}) \;=\;& (\omega + \sigma^{-1})\hat{Y}_t + \frac{1}{2}[-1 - (\bar{\theta}-1)^{-1}\bar{\theta}^{-1}\bar{\theta}' - (\bar{\theta}-1)^{-1}\bar{\theta}^{-1}\bar{\theta}'' - \\
& -2(\bar{\theta}-1)^{-1}\bar{\theta}' + 2\bar{\theta}'\bar{\theta}^{-1}\omega\bar{z}' + \omega\bar{z}' + \tilde{\omega}_1(\bar{z}')^2 + \omega\bar{z}'']E_j\hat{D}_{j,t}^2 + \\
& -\frac{1}{2}(\bar{h}')^{-1}(\bar{h}' + \bar{h}'')(-1 - \bar{\theta}^{-1}(\bar{\theta}-1)^{-1}\bar{\theta}' + \omega\bar{z}')E_j\hat{D}_{j,t}^2 + \\
& +\bar{z}'(-1 - \bar{\theta}^{-1}(\bar{\theta}-1)^{-1}\bar{\theta}' + \omega\bar{z}')E_j\hat{D}_{j,t}^2 + \\
& +\frac{1}{2}(\omega + \sigma^{-1} + \tilde{\omega}_1 + \sigma_1^{-1} + 2\sigma^{-2} + 2\omega\sigma^{-1})\hat{Y}_t^2 + (\omega + \sigma^{-1})\hat{\mu}_t^w\hat{Y}_t \\
& -[\sigma_2^{-1}g_{2,t} + 2\sigma^{-2}g_t + \tilde{\omega}_2 q_{2,t} + \omega\sigma^{-1}g_t + \omega\sigma^{-1}q_t]\hat{Y}_t + \\
& [(1 - \sigma^{-1})\hat{Y}_t + \sigma^{-1}g_t][\hat{\mu}_t^w + \omega_\tau\hat{\tau}_t - \omega q_t - \sigma^{-1}g_t + (\omega + \sigma^{-1})\hat{Y}_t] \\
& +\text{t.i.p.} + \mathcal{O}(||\xi||^3)
\end{aligned}
$$

which can be further simplified to get

$$
\begin{aligned}
b_m\bar{u}_c^{-1}\bar{Y}^{-1}(\tilde{F}_t - \beta\tilde{F}_{t+1}) \;=\;& (\omega + \sigma^{-1})\hat{Y}_t + \frac{1}{2}[-1 - (\bar{\theta}-1)^{-1}\bar{\theta}^{-1}\bar{\theta}' - (\bar{\theta}-1)^{-1}\bar{\theta}^{-1}\bar{\theta}'' + \\
& -2(\bar{\theta}-1)^{-1}\bar{\theta}' + 2\bar{\theta}'\bar{\theta}^{-1}\omega\bar{z}' + \omega\bar{z}' + \tilde{\omega}_1(\bar{z}')^2 + \omega\bar{z}'']E_j\hat{D}_{j,t}^2 + \\
& -\frac{1}{2}(\bar{h}')^{-1}(\bar{h}' + \bar{h}'')(-1 - \bar{\theta}^{-1}(\bar{\theta}-1)^{-1}\bar{\theta}' + \omega\bar{z}')E_j\hat{D}_{j,t}^2 + \\
& +\bar{z}'(-1 - \bar{\theta}^{-1}(\bar{\theta}-1)^{-1}\bar{\theta}' + \omega\bar{z}')E_j\hat{D}_{j,t}^2 + \\
& +\frac{1}{2}(3\omega + 3\sigma^{-1} + \tilde{\omega}_1 + \sigma_1^{-1})\hat{Y}_t^2 + \\
& -[\sigma_2^{-1}g_{2,t} + \tilde{\omega}_2 q_{2,t} + \sigma^{-1}g_t + \omega q_t + (1 - \sigma^{-1})\omega_\tau\hat{\tau}_t - (1 + \omega)\hat{\mu}_t^w]\hat{Y}_t + \\
& +\text{t.i.p.} + \mathcal{O}(||\xi||^3).
\end{aligned}
$$

We can then integrate the above equation starting from period t_0 obtaining

$$b_f \tilde{F}_{t_0} = \bar{u}_y \bar{Y} \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ \hat{Y}_t + \frac{1}{2} b_d E_j \hat{D}_{j,t}^2 + \frac{1}{2} b_{yy} \hat{Y}_t^2 - b_{y\xi} \xi_t \hat{Y}_t \} + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \quad \text{(A.91)}$$

where we have defined $b_f \equiv b_m(\omega + \sigma^{-1})^{-1}$

$$\begin{aligned} b_d \equiv & -(\omega+\sigma^{-1})^{-1}(\bar{h}')^{-1}(\bar{h}'+\bar{h}'')(-1-\bar{\theta}^{-1}(\bar{\theta}-1)^{-1}\bar{\theta}'+\omega\bar{z}')+ \\ & +(\omega+\sigma^{-1})^{-1}[-1-(\bar{\theta}-1)^{-1}\bar{\theta}^{-1}\bar{\theta}'-(\bar{\theta}-1)^{-1}\bar{\theta}^{-1}\bar{\theta}''+ \\ & -2(\bar{\theta}-1)^{-1}\bar{\theta}'+2\bar{\theta}'\bar{\theta}^{-1}\omega\bar{z}'+\omega\bar{z}'+\tilde{\omega}_1(\bar{z}')^2+\omega\bar{z}'']+ \\ & +2(\omega+\sigma^{-1})^{-1}\bar{z}'(-1-\bar{\theta}^{-1}(\bar{\theta}-1)^{-1}\bar{\theta}'+\omega\bar{z}') \end{aligned}$$

$$b_{yy} \equiv (\omega+\sigma^{-1})^{-1}(3\omega+3\sigma^{-1}+\tilde{\omega}_1+\sigma_1^{-1})$$

$$b_{y\xi}\xi_t \equiv (\omega+\sigma^{-1})^{-1}[\sigma_2^{-1}g_{2,t}+\tilde{\omega}_2 q_{2,t}+\sigma^{-1}g_t+\omega q_t-(1+\omega)\hat{\mu}_t^w+\omega_\tau\hat{\tau}_t]$$

Equation (A.91) coincides with equation (4.17) in the main text. In particular we focus on general preferences of the form

$$\tilde{u}(C_t;\xi_t) = \tilde{u}(C_t;\xi_{c,t}), \quad \text{(A.92)}$$

$$\tilde{v}(H_t;\xi_t) = \tilde{v}(H_t;\xi_{h,t}), \quad \text{(A.93)}$$

where $\{\xi_{c,t},\xi_{h,t}\}$ are bounded exogenous scalar disturbance process. In particular for $\tilde{u}(C;\xi_c)$ we assume that $\tilde{\sigma}^{-1} = -\bar{\tilde{u}}_{cc}\bar{C}/\bar{\tilde{u}}_c$, $\bar{c}_t = \tilde{\sigma}\bar{\tilde{u}}_{c\xi}\xi_{c,t}/\bar{\tilde{u}}_c$, $\bar{\sigma}_1^{-1} \equiv -[\bar{\tilde{u}}_{ccc}\bar{C}/\bar{\tilde{u}}_{cc}+1]s_C^{-1}$, $\bar{\sigma}_2^{-1} \equiv -\bar{\tilde{u}}_{cc\xi}\bar{Y}/\bar{\tilde{u}}_{c\xi}$. Substituting $Y_t - G_t$ for C_t and defining $u(Y;\xi) \equiv \tilde{u}(Y-G;\xi_c)$, we obtain that $\sigma^{-1} \equiv -\bar{u}_{yy}\bar{Y}/\bar{u}_y = \tilde{\sigma}^{-1}s_C^{-1}$, $\sigma_1^{-1} \equiv -\bar{u}_{yyy}\bar{Y}^2/\bar{u}_y = -\sigma^{-1}(s_C^{-1}+\bar{\sigma}_1^{-1})$ and that $g_t \equiv \sigma\bar{u}_{y\xi}\xi_t/\bar{u}_y = (\hat{G}_t + s_C\bar{c}_t) + \mathcal{O}(||\xi||^2)$, $g_{2,t} \equiv \sigma_2\bar{u}_{yy\xi}\xi_t\bar{Y}/\bar{u}_y = -\sigma_2\sigma^{-1}s_C^{-1}\hat{G}_t - \sigma_2\sigma^{-1}\bar{\sigma}_2^{-1}g_t + \sigma_2\sigma^{-1}(\bar{\sigma}_2^{-1}-\bar{\sigma}_1^{-1})\hat{G}_t + \mathcal{O}(||\xi||^2)$.

For the function $\tilde{v}(H_t;\xi_t)$, we assume that $\nu \equiv \bar{\tilde{v}}_{hh}\bar{h}/\bar{\tilde{v}}_h$, $\nu_1 \equiv \bar{\tilde{v}}_{hhh}\bar{h}^2/\bar{\tilde{v}}_h$, $\nu_2 \equiv \bar{\tilde{v}}_{hh\xi}\bar{h}/\bar{\tilde{v}}_{h\xi}$, $\bar{h}_t = -\nu^{-1}\bar{\tilde{v}}_{h\xi}\xi_{h,t}/\bar{\tilde{v}}_h$. For the production function $y_t = A_t f(h_t)$, we define $\phi^{-1} \equiv \bar{f}'\bar{h}/\bar{f}$, $\phi_1 \equiv \bar{f}''\bar{h}/\bar{f}'$, $\phi_2 \equiv \bar{f}'''\bar{h}^2/\bar{f}'$. We further define the function $v(y;\xi) \equiv \tilde{v}(f^{-1}(y/A);\xi)$ we obtain that $\omega \equiv \bar{v}_{yy}\bar{Y}/\bar{v}_y = \nu\phi - \phi\phi_1$, $q_t \equiv -\omega^{-1}\bar{v}_{y\xi}\xi_t/\bar{v}_y = \omega^{-1}(\nu\bar{h}_t+\phi(1+\nu)a_t)+\mathcal{O}(||\xi||^2)$; defining $\bar{\omega}_1 \equiv (\nu_1\phi^2-3\nu\phi^2\phi_1-\phi^2\phi_2+3\phi^2\phi_1^2+\omega)/\omega$ we can further obtain that $\tilde{\omega}_1 \equiv \bar{v}_{yyy}\bar{Y}^2/\bar{v}_y = \omega(\bar{\omega}_1-1)$ and that $q_{2,t} \equiv \tilde{\omega}_2^{-1}\bar{v}_{yy\xi}\xi_t\bar{Y}/\bar{v}_y = \tilde{\omega}_2^{-1}(\omega^2 q_t - \tilde{\omega}_3 a_t + \tilde{\omega}_4\bar{h}_t) + \mathcal{O}(||\xi||^2)$ where we have defined $\tilde{\omega}_3 \equiv \omega(1+\omega) - 2\phi(\nu - \phi_1) - \omega_1$, $\tilde{\omega}_4 \equiv \nu\phi(\nu-\nu_2)$. In the example of the text, we still assume Dixit-Stiglitz preferences. In this cases $\psi(x) = x^{\frac{\theta-1}{\theta}}$, $z(x) = x^{-\theta}$, $h(x) = x^{1-\theta}$ and then $\bar{z}' = -\theta$, $\bar{z}'' = \theta(1+\theta)$, $\bar{h}' = 1-\theta$, $\bar{h}'' = -\theta(1-\theta)$, $\bar{\theta} = \theta$, $\bar{\theta}' = \bar{\theta}'' = 0$. It follows that

$$b_{yy} = (2 + \omega - \sigma^{-1}) + \Phi \frac{\sigma^{-1}(1 - s_C^{-1}) + \omega(\bar{\omega}_1 - \omega) - \sigma^{-1}(\bar{\sigma}_1^{-1} - \sigma^{-1})}{(\omega + \sigma^{-1})}$$

$$\begin{aligned} b_{y\xi}\xi_t &= (\omega + \sigma^{-1})^{-1}[\sigma^{-1}(1 - \sigma^{-1})g_t + \sigma^{-1}(\sigma^{-1} - \bar{\sigma}_2^{-1})g_t - \sigma^{-1}s_C^{-1}\hat{G}_t + \sigma^{-1}(\bar{\sigma}_2^{-1} - \bar{\sigma}_1^{-1})\hat{G}_t \\ &\quad \omega(1+\omega)q_t - \tilde{\omega}_3 a_t + \tilde{\omega}_4 \bar{h}_t - (1+\omega)\hat{\mu}_t^w + \omega_\tau(1 - \sigma^{-1})\hat{\tau}_t] + \mathcal{O}(||\xi||^2) \end{aligned}$$

$$b_d = \frac{(1 + \omega\theta)\theta(1+\omega) + \omega(\bar{\omega}_1 - \omega)\theta^2}{(\omega + \sigma^{-1})}$$

From which we can obtain that

$$\begin{aligned} q_{yy} &= a_{yy} + \Phi b_{yy} \\ &= (\omega + \sigma^{-1}) + \Phi(1 - \sigma^{-1}) + \Phi \frac{\sigma^{-1}(1 - s_C^{-1}) + \omega(\bar{\omega}_1 - \omega) - \sigma^{-1}(\bar{\sigma}_1^{-1} - \sigma^{-1})}{(\omega + \sigma^{-1})} \end{aligned}$$

$$\begin{aligned} q_d &= a_d + \Phi b_d \\ &= \frac{\theta(1 + \omega\theta)}{(\omega + \sigma^{-1})}[(\omega + \sigma^{-1}) + \Phi(1 - \sigma^{-1})] + \frac{\Phi\omega(\bar{\omega}_1 - \omega)\theta^2}{(\omega + \sigma^{-1})} \end{aligned}$$

$$\begin{aligned} \hat{Y}_t^* &= q_y^{-1}(a_\xi \xi_t + \Phi_y b_\xi \xi_t) \\ &= q_y^{-1}\{\frac{[(\omega + \sigma^{-1}) + \Phi_y(1 - \sigma^{-1})]}{(\omega + \sigma^{-1})}(\sigma^{-1}g_t + \omega q_t) - \frac{\Phi_y}{(\omega + \sigma^{-1})}[\sigma^{-1}s_C^{-1} - \sigma^{-1}(\bar{\sigma}_2^{-1} - \bar{\sigma}_1^{-1})]\hat{G}_t \\ &\quad + \frac{\Phi_y}{(\omega + \sigma^{-1})}[\sigma^{-1}(\sigma^{-1} - \bar{\sigma}_2^{-1})g_t - (1+\omega)\hat{\mu}_t^w + \omega_\tau(1 - \sigma^{-1})\hat{\tau}_t - \tilde{\omega}_3 a_t + \tilde{\omega}_4 \bar{h}_t] + \mathcal{O}(||\xi||^2) \\ &= [\omega_1 \hat{Y}_t^n - \omega_2 \hat{G}_t + \omega_3 \hat{u}_t^w + \omega_4 \hat{\tau}_t + \omega_5 g_t + \omega_6 a_t + \omega_7 \bar{h}_t] + \mathcal{O}(||\xi||^2) \end{aligned}$$

where the coefficients ω_i are defined in the text.

A.8 Proof of Propositions

PROPOSITION 1. Given Δ_{t_0-1}, let the process $\{x_t\}$ be determined by (i) choosing x_{t_0} and state-contingent commitments $X_{t_0+1}(\xi_{t_0+1})$ to solve the first-stage problem just stated, and (ii) for each possible state of the world ξ_{t_0+1}, choosing the evolution of x_t for $t \geq t_0 + 1$ so as to maximize U_{t_0+1}, among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0 + 1$, given Δ_{t_0}, and that are also consistent with the value of $X_{t_0+1}(\xi_{t_0+1})$ determined in the first stage. Then the process $\{x_t\}$ represents a *Ramsey policy;* that is, it maximizes U_{t_0} among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}.

Proof. Recall that $x_t \equiv (\pi_t, \Delta_t, \hat{Y}_t)$ and $X_t \equiv (F_t, K_t)$. Let $\mathcal{C}$ the set of process $\{x_t,\, X_t\}$ for each $t \geq t_0$ where x_{t_0}, X_{t_0} and X_{t_0+1} are such that

(i) Π_{t_0} and Δ_{t_0} satisfy (1.20);

(ii) the values F_{t_0}

$$F_{t_0} = \hat{F}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}), \tag{A.94}$$

where

$$\hat{F}[x, X(\cdot)](\xi_t) \equiv (1-\tau_t) f(Y_t; \xi_t) + \alpha\beta E_t\{\Pi(F_{t+1}, K_{t+1})^{\theta-1} F_{t+1}\},$$

and K_{t_0}

$$K_{t_0} = \hat{K}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}), \tag{A.95}$$

satisfy

$$\Pi_{t_0} = \Pi(F_{t_0}, K_{t_0}); \tag{A.96}$$

Furthermore the process $\{x_t,\, X_t\}$ for each $t \geq t_0 + 1$ satisfy (1.14), (1.15), (1.19) and (1.20) for each $t \geq t_0 + 1$. Let $\mathcal{R}$ the set of process $\{x_t,\, X_t\}$ for each $t \geq t_0$ that satisfy (1.14), (1.15), (1.19) and (1.20) for each $t \geq t_0$. as in the Ramsey's plan. First, we show that $\mathcal{C} \subset \mathcal{R}$. Let $\{x_t,\, X_t\} \in \mathcal{C}$. It is clear that the process $\{x_t,\, X_t\}$ for each $t \geq t_0 + 1$ satisfy the same constraints, for each $t \geq t_0 + 1$, of the *Ramsey constraints.* In particular (1.14), (1.15) at time $t_0 + 1$ imply

$$F_{t_0+1} \equiv E_{t_0+1} \sum_{T=t_0+1}^{\infty} (\alpha\beta)^{T-t_0-1} (1-\tau_T) f(Y_T; \xi_T) \left(\frac{P_T}{P_{t_0+1}}\right)^{\theta-1}, \tag{A.97}$$

$$K_{t_0+1} \equiv E_{t_0+1} \sum_{T=t_0+1}^{\infty} (\alpha\beta)^{T-t_0-1} k(Y_T;\xi_T) \left(\frac{P_T}{P_{t_0+1}}\right)^{\theta(1+\omega)}. \tag{A.98}$$

Since F_{t_0} and K_{t_0} satisfy (A.94) and (A.95), respectively, it follows that

$$\begin{aligned} F_{t_0} &= (1-\tau_{t_0}) f(Y_{t_0};\xi_{t_0}) + \alpha\beta E_{t_0}\{\Pi_{t_0+1} F_{t_0+1}\} \\ &= (1-\tau_{t_0}) f(Y_{t_0};\xi_{t_0}) + \alpha\beta E_{t_0}\left\{\Pi_{t_0+1} E_{t_0+1} \sum_{T=t_0+1}^{\infty} (\alpha\beta)^{T-t_0-1} (1-\tau_T) f(Y_T;\xi_T) \left(\frac{P_T}{P_{t_0+1}}\right)^{\theta-1}\right\} \\ &= E_{t_0} \sum_{T=t_0}^{\infty} (\alpha\beta)^{T-t_0} (1-\tau_T) f(Y_T;\xi_T) \left(\frac{P_T}{P_{t_0}}\right)^{\theta-1}, \end{aligned}$$

and

$$\begin{aligned} K_{t_0} &= k(Y_{t_0};\xi_{t_0}) + \alpha\beta E_{t_0}\{\Pi_{t_0+1}^{\theta(1+\omega)} K_{t_0+1}\} \\ &= k(Y_{t_0};\xi_{t_0}) + \alpha\beta E_{t_0}\left\{\Pi_{t_0+1}^{\theta(1+\omega)} E_{t_0+1} \sum_{T=t_0+1}^{\infty} (\alpha\beta)^{T-t_0-1} k(Y_T;\xi_T) \left(\frac{P_T}{P_{t_0+1}}\right)^{\theta(1+\omega)}\right\} \\ &= E_{t_0} \sum_{T=t_0}^{\infty} (\alpha\beta)^{T-t_0} k(Y_T;\xi_T) \left(\frac{P_T}{P_{t_0}}\right)^{\theta(1+\omega)}, \end{aligned}$$

where we have used (A.97),(A.98) and the definition of Π. It follows that constraints (1.14), (1.15) are also satisfied at time t_0. Moreover Π_{t_0} and Δ_{t_0} satisfy (1.20) because of point (i) above and F_{t_0}, K_{t_0} and Π_{t_0} satisfy (1.19) at time t_0 because of point (ii) above. It follows that $\{x_t,\ X_t\} \in \mathcal{R}$. Moreover we can further show that $\mathcal{R} \subset \mathcal{C}$ following the same the construction given in the text. It follows that $\mathcal{R} \subseteq \mathcal{C}$ and $\mathcal{R} \supseteq \mathcal{C}$.

Given Δ_{t_0-1}, let the process $\{x_t,\ X_t\} \in \mathcal{C}$ be determined by (i) choosing x_{t_0} and state-contingent commitments $X_{t_0+1}(\xi_{t_0+1})$ to solve the first-stage problem in the text, and (ii) for each possible state of the world ξ_{t_0+1}, choosing the evolution of x_t, X_{t+1} for $t \geq t_0+1$ so as to maximize U_{t_0+1}, among all of the paths consistent with (1.14), (1.15), (1.19) and (1.20) for each $t \geq t_0+1$, given Δ_{t_0}, and that are also consistent with the value of $X_{t_0+1}(\xi_{t_0+1})$ determined in the first stage. Let instead $\{x_t^*,\ X_t^*\} \in \mathcal{R}$ the *Ramsey policy* that maximizes U_{t_0} among all of the paths consistent with (1.14), (1.15), (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}. Let $U_{t_0}^*$ the utility under the *Ramsey policy*. From the previous analysis it follows that $\{x_t,\ X_t\} \in \mathcal{R}$; let $U_{t_0}^c$ the utility evaluated at the process $\{x_t,\ X_t\}$ then $U_{t_0}^c \leq U_{t_0}^*$. Note that

$\max \hat{J}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}) \geq U_{t_0}$ for all processes in $\mathcal{C}$ and in particular for $\{x_t^*,\ X_t^*\}$ since $\mathcal{R} \subset \mathcal{C}$. Then it should be that $\max \hat{J}[x_{t_0}, X_{t_0+1}(\cdot)](\xi_{t_0}) = U_{t_0}^c = U_{t_0}^*$. It follows that the process $\{x_t,\ X_t\}$ is a *Ramsey policy.*

PROPOSITION 2. Given some $(\Delta_{t_0-1}, X_{t_0}) \in \mathcal{F}(t_0)$, consider the sequential decision problem in which in each period $t \geq t_0$, $(x_t, X_{t+1}(\cdot))$ are chosen to maximize $\hat{J}[x_t, X_{t+1}(\cdot)](\xi_t)$, subject to constraints (i) – (iii) of the "first stage" problem stated above, given the predetermined state variable Δ_{t-1} and the precommitted values X_t. Then the process $\{x_t\}$ that is chosen in this way is the process that maximizes U_{t_0} among all of the paths consistent with (1.19) and (1.20) for each $t \geq t_0$, given Δ_{t_0-1}, and also consistent with the specified values X_{t_0}.

Proof. Similar to proposition 1.

PROPOSITION 3. Randomization of monetary policy reduces the expected losses (2.12) — and hence is locally welfare-reducing in the exact problem as well — if and only if the quadratic form (3.2) is positive definite. Furthermore, if and only if this is true, processes $\{\pi_t, \hat{Y}_t\}$ that satisfy the first-order conditions for the LQ optimization problem [discussed further below] represent a loss minimum, and hence an approximation to (at least a local) welfare maximum in the exact problem.

Furthermore, the necessary and sufficient conditions for (3.2) to be negative definite reduce to the following: q_π and q_y are not *both* equal to zero; and *either* (i) $q_y \geq 0$ and

$$q_\pi + (1 - \beta^{1/2})^2 \kappa^{-2} q_y \geq 0, \tag{A.99}$$

holds, or (ii) $q_y \leq 0$ and

$$q_\pi + (1 + \beta^{1/2})^2 \kappa^{-2} q_y \geq 0, \tag{A.100}$$

holds.

Proof. Consider the possible benefits of a dynamic response of inflation and output to a sunspot shock v_t at date t. It is assumed that v_t is distributed with a mean of zero, variance σ_v^2, independent of all other disturbances (fundamental shocks, or other sunspots), and is revealed at date t. Suppose that the occurrence of this shock adds a contribution $\varphi_j v_t$ to π_{t+j}, for each $j \geq 0$, where $\{\varphi_j\}$ is an arbitrary bounded sequence of coefficients. (This is the effect of the sunspot in the linear approximation to the solution which suffices for our computation of a second-order approximation to welfare.) It then follows from (2.14) that the shock also adds a contribution $\kappa^{-1}(\varphi_j - \beta\varphi_{j+1})v_t$ to $\hat{Y}_{t+j}$ for each $j \geq 0$.

The response to this sunspot thus adds terms to the loss measure (2.12) equal to

$$\beta^t \sigma_v^2 \sum_{j=0}^{\infty} \beta^j \left[q_\pi \varphi_j^2 + q_y \left(\frac{\varphi_j - \beta\varphi_{j+1}}{\kappa} \right)^2 \right]. \tag{A.101}$$

Randomization of monetary policy is (locally) undesirable if and only if this expression is positive in the case of all possible non-zero bounded sequences $\{\varphi_j\}$. This is turn holds if and only if the quadratic form

$$||\varphi|| \equiv \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Lambda_{ij} \varphi_i \varphi_j \tag{A.102}$$

is *positive definite,* where

$$\Lambda_{00} = a + b \equiv q_\pi + \frac{q_y}{\kappa^2},$$

$$\Lambda_{ii} = \beta^i a \equiv \beta^i \left[q_\pi + (1+\beta)\frac{q_y}{\kappa^2} \right] \qquad \text{for each} \quad i \geq 1,$$

$$\Lambda_{i,i+1} = \beta^i b \equiv -\beta^{i+1} \frac{q_y}{\kappa^2} \qquad \text{for each} \quad i \geq 0,$$

$$\Lambda_{i,i-1} = \Lambda_{i-1,i} \qquad \text{for each} \quad i \geq 1,$$

and

$$\Lambda_{i,j} = 0 \qquad \text{if} \quad |j - i| > 1.$$

Let us introduce the change of variables

$$\tilde{\varphi}_j \equiv \varphi_j + \frac{b}{\tilde{\Lambda}_j} \varphi_{j+1}$$

for each $j \geq 0$, where the sequence of coefficients $\tilde{\Lambda}_j$ is defined by the recursion

$$\tilde{\Lambda}_{j+1} = a - \frac{b^2}{\beta \tilde{\Lambda}_j} \tag{A.103}$$

for each $j \geq 0$, starting from the initial value

$$\tilde{\Lambda}_0 = \Lambda_{00} = a + b. \tag{A.104}$$

We then can equivalently express (A.102) as

$$||\varphi|| \equiv \sum_{j=0}^{\infty} \beta^j \tilde{\Lambda}_j \tilde{\varphi}_j^2$$

which is obviously positive definitive if and only if $\widetilde{\Lambda}_j > 0$ for all $j \geq 0$..

We thus wish to consider whether the law of motion (A.103) defines a sequence that is positive for all $j \geq 0$, starting from the initial condition (A.104). In the case that $b = 0$, (A.103) obviously implies that $\tilde{\Lambda}_j = a$ for all j, so that $||\varphi||$ is positive definite if and only if $a > 0$. We turn now to the case in which $b \neq 0$.

Let us consider the graph of the right-hand side of (A.103), for positive values of $\tilde{\Lambda}_j$. This is a monotonically increasing function, that takes a positive value less than $\tilde{\Lambda}_j$ for all $\tilde{\Lambda}_j$ large enough; thus the graph is in the positive orthant, but below the diagonal, for all large enough $\tilde{\Lambda}_j$. If the function is below the diagonal for all $\tilde{\Lambda}_j > 0$, then (A.103) implies that the sequence $\{\tilde{\Lambda}_j\}$, if initially positive-valued, must be monotonically decreasing until it eventually becomes negative-valued for some finite j.[44] Hence $\tilde{\Lambda}_j > 0$ for all j requires that the graph of the right-hand side of (A.103) cross the diagonal at some positive value.

This occurs if and only if the quadratic equation

$$\Lambda^2 - a\Lambda + \beta^{-1}b^2 = 0 \tag{A.105}$$

has two real roots, and at least the larger root is positive. Because the product of the roots must equal $\beta^{-1}b^2 > 0$, in this case both roots must be positive. Then the dynamics implied by (A.103) will still imply that $\tilde{\Lambda}_j < 0$ for some finite j if the initial value $\tilde{\Lambda}_0$ is less than the smaller root. Hence $||\varphi||$ is positive definite if and only if (i) the quadratic equation (A.105) has two positive real roots (not necessarily distinct), and (ii) the initial value $\tilde{\Lambda}_0$ specified in (A.104) is not less than the smaller root.

Condition (i) holds if and only if

$$a^2 \geq 4\beta^{-1}b^2 \tag{A.106}$$

and

$$a > 0. \tag{A.107}$$

[44]Here we note that the function must become unboundedly negative as $\tilde{\Lambda}_j$ approaches zero. Thus we may neglect the possibility of a graph that crosses the diagonal exactly at zero.

(Here (A.106) is necessary and sufficient for the existence of two real roots – necessarily of the same sign – which are then both positive if and only if (A.107) holds as well.) Given (A.107), we may equivalently write (A.106) as

$$a \geq 2\beta^{-1/2}|b| \tag{A.108}$$

When condition (i) holds, condition (ii) holds as well if and only if

$$a + b \geq \frac{1}{2}\left\{a - [a^2 - 4\beta^{-1}b^2]^{1/2}\right\},$$

or equivalently,

$$a + 2b + [a^2 - 4\beta^{-1}b^2]^{1/2} \geq 0. \tag{A.109}$$

However, (A.107) – (A.108) imply that $a > 2|b|$, and hence that $a + 2b > 0$. Thus (A.109) is guaranteed by the previous two equations. Hence when $b \neq 0$, (A.107) and (A.108) are necessary and sufficient for $||\varphi||$ to be positive definite. We have also seen that if $b = 0$, (A.107) is necessary and sufficient, while (A.108) is implied in this case by (A.107). Hence we can state with full generality that (A.107) and (A.108) are both necessary and sufficient for the quadratic form $||\varphi||$ to be positive definite, and hence for randomization of monetary policy to be (locally) welfare-reducing.

In our application, (A.107) holds if and only if

$$q_\pi + (1 + \beta)\kappa^{-2}q_y > 0. \tag{A.110}$$

Condition (A.108) holds if and only *either* (i) $q_y \geq 0$ and

$$q_\pi + (1 - \beta^{1/2})^2\kappa^{-2}q_y \geq 0, \tag{A.111}$$

or (ii) $q_y \leq 0$ and

$$q_\pi + (1 + \beta^{1/2})^2\kappa^{-2}q_y \geq 0. \tag{A.112}$$

We note further that except in the case that $q_\pi = q_y = 0$, (A.111) implies (A.110) under the hypothesis that $q_y \geq 0$, and similarly (A.112) implies (A.110) under the hypothesis that $q_y \leq 0$. Hence the necessary and sufficient conditions for randomization of monetary policy to be (locally) welfare-reducing reduce to the following: q_π and q_y are not *both* equal to zero; and *either* (i) $q_y \geq 0$ and (A.111) holds, or (ii) $q_y \leq 0$ and (A.112) holds.

Conversely, (some appropriately chosen type of) randomization of monetary policy is welfare-improving if and only if *either* (i) $q_y \geq 0$ and (A.111) fails to hold (i.e., the strict inequality with the sign reversed holds), or (ii) $q_y \leq 0$ and (A.112) fails to hold.

Using the expressions for q_π and q_y we can then derive inequalities that the model parameters must satisfy in order for randomization not to be locally welfare-improving.

Consider the optimization problem of choosing $\{\pi_t, \hat{Y}_t\}$ for each $t \geq t_0$ that minimize

$$L_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2} \pi_t^2 + \frac{q_y}{2} (\hat{Y}_t - \hat{Y}_t^*)^2 \right\}. \tag{A.113}$$

under the constraints implied by the linear structural relation (2.11) holding in each period $t \geq t_0$ and subject also to the constraints that a certain predetermined value for $\pi_{t_0} = \bar{\pi}_{t_0}$ be achieved. Substituting the constraints into (A.113), we obtain that

$$\begin{aligned} L_{t_0} &= \frac{q_\pi}{2} \bar{\pi}_{t_0}^2 + \frac{q_y}{2\kappa^2} (\bar{\pi}_{t_0} - u_{t_0} - \beta E_{t_0} \pi_{t+1})^2 + \\ &\quad + E_{t_0} \sum_{t=t_0+1}^{\infty} \beta^{t-t_0} \left\{ \frac{q_\pi}{2} \pi_t^2 + \frac{q_y}{2\kappa^2} (\pi_t - u_t - \beta E_t \pi_{t+1})^2 \right\}. \end{aligned}$$

Taking the first-order conditions with respect to the sequence $\{\pi_t\}$ for each $t \geq t_0 + 1$, we obtain that

$$\frac{\partial L_{t_0}}{\partial \pi_t} = E_{t_0} \beta^{t-t_0} \left\{ -\frac{q_y}{\kappa^2} \beta (\pi_{t-1} - u_{t-1} - \beta E_{t-1} \pi_t) + \beta q_\pi \pi_t + \beta \frac{q_y}{\kappa^2} (\pi_t - u_t - \beta E_t \pi_{t+1}) \right\} = 0.$$

Second-order conditions are

$$\frac{\partial^2 L_{t_0}}{\partial^2 \pi_t} = \beta^{t+1-t_0} \left[\frac{q_y}{\kappa^2} (1+\beta) + q_\pi \right]$$

and

$$\frac{\partial^2 L_{t_0}}{\partial^2 \pi_t \pi_{t+1}} = \beta^{t+1-t_0} \left[-\beta \frac{q_y}{\kappa^2} \right]$$

for each $t \geq t_0 + 1$. In particular the set of second-order conditions can be collected in a quadratic form

$$||\varphi^c|| \equiv \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Lambda_{ij}^c \varphi_i^c \varphi_j^c$$

where $\Lambda_{ij}^c = \Lambda_{ij}$ for all $i > 0$ and $j > 0$ and $\Lambda_{00}^c = \frac{q_y}{\kappa^2}(1+\beta) + q_\pi$ for bounded variables φ_j^c, φ_i^c. The allocation that satisfy the first-order conditions is a loss minimum if and only if $||\varphi^c||$ is positive definite. It is easy to see that even under the initial condition Λ^c_{00} the conditions stated above applies also to this quadratic form $||\varphi^c||$. It follows that the conditions under which the processes $\{\pi_t, \hat{Y}_t\}$ that satisfy the first-order conditions for the LQ optimization problem represent a loss minimum coincide with the conditions under which randomization is welfare decreasing.

PROPOSITION 4. Suppose that $\hat{Y}^n_t = \hat{Y}^*_t$ at all times, and that the conditions stated in Proposition 3 are satisfied. Then the policy that uniquely minimizes (2.12) is the one under which $\pi_t = 0$ at all times, regardless of the realizations of the exogenous disturbances [as long as these are small enough to make such an equilibrium possible].

Proof. If $\hat{Y}^n_t = \hat{Y}^*_t$ at all times, then $u_t = 0$ at all times. First-order conditions for an optimum are

$$q_\pi \pi_t + \varphi_t - \varphi_{t-1} = 0,$$

$$q_y x_t - \kappa \varphi_t = 0,$$

$$\pi_t - \kappa x_t - \beta E_t \pi_{t+1} = 0.$$

By assuming that disturbances are small enough, a policy of $\pi_t = 0$ at all times is feasible since the nominal interest rate required for this equilibrium is non-negative at all times. Moreover it satisfies the above necessary condition for an optimum, since $x_t = 0$ and $\varphi_t = 0$ at all times provided $\varphi_{t_0-1} = 0$. Proposition 3 assures that second-order conditions are satisfied and that $\pi_t = 0$ is a unique mininum for all $t \geq t_0$ provided $\bar{\pi}_{t_0}$ is chosen such that $\bar{\pi}_{t_0} = 0$.

PROPOSITION 5. Consider a model with the isoelastic functional forms (1.3) – (1.4), and parameter values $\omega \geq 0, \sigma^{-1} > 0$, and suppose that there are random fluctuations in the composite disturbance term $\omega q_t + \sigma^{-1}\bar{c}_t$. [This is generally true if either preferences or technology are random.] Then $\hat{Y}^n_t = \hat{Y}^*_t$ at all times — so that the "cost-push" term in the aggregate-supply relation (2.14) is zero at all times — if and only if (i) there are no random variations in the wage markup or the tax rate ($\hat{\mu}^w_t = \hat{\tau}_t = 0$ at all times); and (ii) *either* (a) the steady-state level of output is efficient ($\Phi = 0$) or (b) there are no government purchases ($G_t = 0$ at all times).

Proof. In the text.

Proposition 6. Suppose that $q_\pi \neq 0$, and that (3.11) is satisfied in addition to the conditions listed in Proposition 3. Then in the case of any small enough value of $\bar{\pi}_{t_0}$, and any sufficiently tightly bounded fluctuations in the cost-push disturbance process $\{u_t\}$, the solution to the optimization problem stated in Proposition 2 involves fluctuations $\{\pi_t, x_t\}$ that remain forever within any given neighborhood of the steady-state values $(0, 0)$. These optimal dynamics are furthermore approximated (arbitrarily well, in the case of tight enough bounds on $\bar{\pi}_{t_0}$ and on the amplitude of the cost-push terms) by the log-linear dynamics corresponding to the unique bounded solution to equations (2.14), (3.6) and (3.7) consistent with initial condition (2.13).

This solution is obtained by solving (3.6) and (3.7) for π_t and x_t respectively, where once again the multiplier process $\{\varphi_t\}$ is specified recursively by the relation (3.12). In this latter expression, as before, μ is the root of (3.10) that satisfies $-1 < \mu < 1$, and the initial value φ_{t_0-1} is chosen so that that the solution is consistent with (2.13).

Proof. [TO BE DONE]

Proposition 7. Given some $(D_{t_0-1},\ F_{t_0}) \in \mathcal{F}$ consider the sequential decision problem in which each period $t \geq t_0$, $(x_t, F_{t+1}(\cdot))$ are chosen to maximize $\hat{J}[x, F(\cdot)](\xi_t)$, subject to constraints $(i) - (iii)$ stated above, given the predetermined state variable D_{t-1} and the precommitted values F_t. Then the process $\{x_t\}$ that is chosen in this way is the process that maximizes U_{t_0} among all of the paths consistent with (4.8) and (4.11) for each $t \geq t_0$, given D_{t_0-1} and also consistent with the specified values F_{t_0}.

Proof. Similar to Proposition 1.

References

[1] Benigno, Pierpaolo, and Michael Woodford (2003a), "Optimal Monetary and Fiscal Policy a Linear-Quadratic Approach," in M. Gertler and K. Rogoff, eds., NBER Macroeconomics Annual 2003, vol. 18, forthcoming.

[2] Benigno, Pierpaolo, and Michael Woodford (2003b), "Optimal Policy under Recursive Commitment: A Linear-Quadratic Approach ," in preparation.

[3] Calvo, Guillermo (1983), "Staggered Prices in a Utility-Maximizing Framework," *Journal of Monetary Economics*, 12: 383-398.

[4] Clarida, Richard, Jordi Gali, and Mark Gertler (1999), "The Science of Monetary Policy: a New Keynesian Perspective," *Journal of Economic Literature*, 37 (4), pp. 1661-1707.

[5] Dupor, William (2003), "Optimal Random Monetary Policy with Nominal Rigidity," *Journal of Economic Theory*, 112 (1): 66-78.

[6] Giannoni, Marc, and Michael Woodford (2002), "Optimal Interest-Rate Rules: I. General Theory,"NBER working paper no. 9419, December.

[7] Giannoni, Marc, and Michael Woodford (2003), "Optimal Inflation Targeting Rules,"in B.S. Bernanke and M. Woodford, eds., *The Inflation Targeting Debate,* Chicago: University of Chicago Press, forthcoming.

[8] Goodfriend, Marvin, and Robert King (1997), "The New Neoclassical Synthesis and the Role of Monetary Policy," in B.S. Bernanke and J.J. Rotemberg (eds.), *NBER Macroeconomic Annual 1997*, Cambridge, MA: MIT press.

[9] Khan, Aubhik, Robert G. King, and Alexander L. Wolman (2003), "Optimal Monetary Policy," *Review of Economic Studies* 70(4): 825-860.

[10] Kim, Jinill, Sunghyun Kim, Ernst Schaumburg, and Christopher A. Sims (2003), "Calculating and Using Second Order Accurate Solutions of Discrete Time Dynamic Equilibrium Models ," unpublished manuscript, University of Virginia, June.

[11] King, Robert G., and Alexander L. Wolman (1999), "What Should the Monetary Authority Do when Prices are Sticky," in J.B. Taylor, ed., *Monetary Policy Rules*, Chicago: University of Chicago Press.

[12] Rotemberg, Julio J., and Michael Woodford (1997), "An Optimization-Based Econometric Framework for the Evaluation of Monetary Policy", in B.S. Bernanke and Rotemberg (eds.), *NBER Macroeconomic Annual 1997*, Cambridge, MA: MIT Press. 297-346.

[13] Sutherland, Alan (2002), "A Simple Second-Order Solution Method for Dynamic General Equilibrium Models," unpublished manuscript, University of St. Andrews.

[14] Walsh, Carl (2003), *Monetary Theory and Policy*, 2nd edition, Cambridge, MA:MIT Press.

[15] Woodford, Michael (1999), "Commentary: How Should Monetary Policy Be Conducted in an Era of Price Stability?"in Federal Reserve Bank of Kansas City, *New Challenges for Monetary Policy*.

[16] Woodford, Michael (2001), "Fiscal Requirements for Price Stability,"*Journal of Money, Credit and Banking* 33: 669-728.

[17] Woodford, Michael (2003), *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press.