Technical appendix to "Designing Targeting Rules for International Monetary Policy Cooperation" by Gianluca Benigno and Pierpaolo Benigno

Derivation of the quadratic loss functions, equations (3.21), (3.32), (3.33) in the main text.

In this appendix, we show how to derive a second-order approximation to the sum of the utilities of the consumers belonging to each country (the objective function of the policymaker) which results in a quadratic form and can be correctly evaluated by a log-linear approximation to the structural equilibrium conditions. The method used here follows Benigno and Woodford (2003). In particular we are going to derive equations (3.21), (3.32), (3.33) in the main text.

First we recall that each individual has an utility function of the form

$$U_t^j = \mathrm{E}_t \left\{ \sum_{T=t}^{\infty} \beta^{T-t} \left[U(C_T^j) - V(y_T(j), \xi_T^i) \right] \right\},$$

where the index j denotes a variable that is specific to household j and the index i denotes a variable specific to the country H or F in which j resides. We assume the following functional forms

$$U(C_t^j) \equiv \frac{(C_t^j)^{1-\rho}}{1-\rho}, \qquad V(y_t(j), \xi_t^i) \equiv (a_t^i)^{-\eta} \frac{(y_t(j))^{1+\eta}}{1+\eta}. \tag{1}$$

The objective function of the monetary policymaker of country H is to maximize the sum of the utilities of its consumers given by

$$W = \mathrm{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[U(C_t) - n^{-1} \int_0^n V(y_t(h), \xi_t) dh \right] \right\}, \tag{2}$$

since $C_t^j = C_t$ for all j belonging to each country because of the complete-market assumption.

The objective of the policymaker of country F is

$$W^* = \mathrm{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[U(C_t^*) - (1-n)^{-1} \int_n^1 V(y_t(f), \xi_t^*) dh \right] \right\}. \tag{3}$$

All the variables have the same definitions as in the main text. We further define the inefficient wedges, μ_t and μ_t^*, as a combinations of the mark-ups and the distorting taxes in the following way

$$\frac{1}{\mu_t} \equiv \frac{(1-\tau_t)(\sigma-1)}{\sigma} \qquad \frac{1}{\mu_t^*} \equiv \frac{(1-\tau_t^*)(\sigma-1)}{\sigma}.$$

We approximate the model around a steady state in which the three pairs of exogenous variables $(a_t,\ a_t^*)$, $(G_t,\ G_t^*)$, $(\mu_t,\ \mu_t^*)$ all take constant values equal across countries and such that $\bar{a}$, $\bar{G} > 0$ and $\bar{\mu} \geq 1$ at all times. We further focus on a steady-state in which $\Pi_{H,t} \equiv P_{H,t}/P_{H,t-1} = 1$ and $\Pi_{F,t}^* \equiv P_{F,t}^*/P_{F,t-1}^* = 1$ at all times. The risk-sharing condition implies that

$$U_C(C_t) = U_C(C_t^*)$$

and in the steady state $\bar{C} = \bar{C}^*$. Given that $\Pi_{H,t} = 1$ and $\Pi_{F,t}^* = 1$ in the steady state, the countries' real marginal costs are constant and such that

$$\frac{U_C(\bar{C})}{\bar{\mu}}\bar{p}_H = V_y\left(\bar{p}_H^{-\theta}\bar{C} + \bar{G}, \bar{\xi}\right) \tag{4}$$

for country H and

$$\frac{U_C(\bar{C})}{\bar{\mu}}\bar{p}_F = V_y\left(\bar{p}_F^{-\theta}\bar{C} + \bar{G}, \bar{\xi}\right) \tag{5}$$

for country F, where $p_H \equiv P_H/P$ and $p_F \equiv P_F/P$. We note that given the definition of the general price index we can write

$$1 = n\bar{p}_H^{1-\theta} + (1-n)\bar{p}_F^{1-\theta}. \tag{6}$$

Given the functional forms assumed, equations (4), (5) and (6) imply that $\bar{p}_H = \bar{p}_F = 1$ and that $\bar{Y}_H = \bar{Y}_F^*$. As well $\bar{T} = 1$, where $\bar{T} \equiv \bar{p}_F/\bar{p}_H$. We further note that unless $\bar{\mu} = 1$, the steady-state output and consumption are inefficiently low. For later use, we define $s_c \equiv \bar{C}/\bar{Y}_H = \bar{C}^*/\bar{Y}_F^*$.

First we consider the welfare of the consumers in the home economy and take a second-order approximation to its elements. A second-order approximation to $U(C_t)$ around the above defined steady state yields to

$$U(C_t) = \bar{U}_C\bar{C}[\hat{C}_t + \frac{1}{2}(1-\rho)\hat{C}_t^2] + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \tag{7}$$

where t.i.p. denotes terms that are independent of policy and $\mathcal{O}(||\xi||^3)$ denotes terms that are of third order or higher in the norm of the shocks. Here and in what follows hats variables denote log-deviation of the variable from the steady state, e.g. $\hat{C}_t \equiv \ln C_t/\bar{C}$. A second order expansion to the term $V(y_t(h), \xi_t)$ yields to

$$V(y_t(h), \xi_t) = \bar{V}_y\bar{Y}[\hat{y}_t(h) + \frac{1}{2}(1+\eta)\hat{y}_t^2(h) - \eta\hat{a}_t\hat{y}_t(h)] + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \tag{8}$$

From (8), we can obtain that

$$\frac{\int_0^n V(y_t(h), \xi_t)dh}{n} = \bar{V}_y\bar{Y}[\hat{Y}_{H,t} + \frac{1}{2}(1+\eta)\hat{Y}_{H,t}^2 - \eta\hat{a}_t\hat{Y}_{H,t} + \frac{1}{2}(\sigma^{-1}+\eta)\text{var}_h\hat{y}_t(h)] + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \tag{9}$$

where following Woodford (2003, ch. 6) we have defined

$$E_h \hat{y}_t(h) \equiv n^{-1} \int_0^n \hat{y}_t(h) dh,$$

and used the following relations

$$E_h[\hat{y}_t(h)]^2 = var_h \hat{y}_t(h) + [E_h \hat{y}_t(h)]^2, \tag{10}$$

$$\hat{Y}_{H,t} = E_h \hat{y}_t(h) + \frac{1}{2}\left(\frac{\sigma - 1}{\sigma}\right) var_h \hat{y}_t(h) + \mathcal{O}(||\xi||^3). \tag{11}$$

Equation (11) is derived from a second-order expansion of the output aggregator:

$$Y_H \equiv \left[\left(\frac{1}{n}\right) \int_o^n y(h)^{\frac{\sigma-1}{\sigma}} dh\right]^{\frac{\sigma}{\sigma-1}}.$$

Using the steady-state relations, we can combine (8) and (9) to get the utility flow at time t

$$\begin{aligned} w_t &= \bar{U}_C \overline{C}[\hat{C}_t + \frac{1}{2}(1-\rho)\hat{C}_t^2 - s_c^{-1}\bar{\mu}^{-1}\hat{Y}_{H,t} + \\ &\quad -\frac{1}{2}s_c^{-1}\bar{\mu}^{-1}(1+\eta)\hat{Y}_{H,t}^2 + s_c^{-1}\bar{\mu}^{-1}\eta\hat{a}_t\hat{Y}_{H,t} + \\ &\quad -\frac{1}{2}s_c^{-1}\bar{\mu}^{-1}(\sigma^{-1}+\eta) \cdot \mathrm{var}_h\hat{y}_t(h)] + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{12}$$

where

$$w_t \equiv U(C_t) - \frac{\int_0^n V(y_t(h), \xi_t) dh}{n}.$$

We can then plug (12) into (2) and obtain that a second-order approximation to the welfare criterion for the home country can be written as

$$\begin{aligned} W &= \bar{U}_C \overline{C} E_0\{\sum_{t=0}^{\infty} \beta^t[\hat{C}_t + \frac{1}{2}(1-\rho)\hat{C}_t^2 - s_c^{-1}\bar{\mu}^{-1}\hat{Y}_{H,t} + \\ &\quad -\frac{1}{2}s_c^{-1}\bar{\mu}^{-1}(1+\eta)\hat{Y}_{H,t}^2 + s_c^{-1}\bar{\mu}^{-1}\eta\hat{a}_t\hat{Y}_{H,t} + \\ &\quad -\frac{1}{2}s_c^{-1}\bar{\mu}^{-1}\sigma k^{-1}\pi_{H,t}^2] + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{13}$$

where following Woodford (2003, ch. 6) we have used the fact that

$$\sum_{t=0}^{\infty} \beta^t \mathrm{var}_h \hat{y}_t(h) = \frac{1}{k(1+\sigma\eta)} \sigma^2 \sum_{t=0}^{\infty} \beta^t \pi_{H,t}^2 + \text{t.i.p.} + \mathcal{O}(||\xi||^3),$$

for

$$k \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\frac{1}{(1+\sigma\eta)}.$$

We can write (13) in a vector-matrix notation as

$$W = \bar{U}_C \overline{C} E_0 \{ \sum_{t=0}^{\infty} \beta^t [z_x' x_t - \frac{1}{2} x_t' Z_x x_t - x_t' Z_\xi \xi_t - \frac{1}{2} z_{\pi_h} \pi_{H,t}^2] + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \quad (14)$$

where

$$x_t' \equiv [\hat{Y}_{H,t} \;\; \hat{C}_t \;\; \hat{p}_{H,t} \;\; \hat{Y}_{F,t}^* \;\; \hat{C}_t^* \;\; \hat{p}_{F,t} \;\; \hat{T}_t],$$

$$\xi_t' \equiv [\hat{a}_t \;\; \hat{\mu}_t \;\; \hat{G}_t \;\; \hat{a}_t^* \;\; \hat{\mu}_t^* \;\; \hat{G}_t^*],$$

$$z_x' \equiv [-\bar{\mu}^{-1} s_c^{-1} \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0],$$

$$Z_x \equiv \begin{bmatrix} \bar{\mu}^{-1} s_c^{-1}(1+\eta) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(1-\rho) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Z_\xi \equiv \begin{bmatrix} -\bar{\mu}^{-1} s_c^{-1} \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$z_{\pi_h} \equiv s_c^{-1} \bar{\mu}^{-1} \sigma k^{-1},$$

where

$$k^* \equiv \frac{(1-\alpha^*)(1-\alpha^*\beta)}{\alpha^*}\frac{1}{(1+\sigma\eta)}.$$

We can repeat the same steps and obtain a second-order approximation to the welfare of country F as

$$\begin{aligned} W^* \;\; = \;\; & \bar{U}_C \overline{C} E_0 \{ \sum_{t=0}^{\infty} \beta^t [\hat{C}_t^* + \frac{1}{2}(1-\rho)(\hat{C}_t^*)^2 - s_c^{-1} \bar{\mu}^{-1} \hat{Y}_{F,t}^* + \\ & -\frac{1}{2} s_c^{-1} \bar{\mu}^{-1}(1+\eta)[\hat{Y}_{F,t}^*]^2 + s_c^{-1} \bar{\mu}^{-1} \eta \hat{a}_t^* \hat{Y}_{F,t}^* + \\ & -\frac{1}{2} s_c^{-1} \bar{\mu}^{-1} \sigma \left(k^*\right)^{-1} \pi_{F,t}^{*2}] + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned} \quad (15)$$

which can be re-written in a compact form as

$$W^* = \bar{U}_C \overline{C} E_0 \{ \sum_{t=0}^{\infty} \beta^t [z_x^{*\prime} x_t - \frac{1}{2} x_t' Z_x^* x_t - x_t' Z_\xi^* \xi_t - \frac{1}{2} z_{\pi_f}^* \pi_{F,t}^{*2}] + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \quad (16)$$

where

$$z_x^{*\prime} \equiv [0 \;\; 0 \;\; 0 \;\; -\bar{\mu}^{-1} s_c^{-1} \;\; 1 \;\; 0 \;\; 0],$$

$$Z_x^* \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\mu}^{-1} s_c^{-1}(1+\eta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(1-\rho) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Z_\xi^* \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\mu}^{-1} s_c^{-1} \eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$z_{\pi_f}^* \equiv s_c^{-1} \bar{\mu}^{-1} \sigma \left(k^*\right)^{-1}.$$

We now derive a second-order approximation to the structural equilibrium conditions. Let us focus on country H. As shown in the text (equation 1.5), the first-order condition for sellers that can reset their price at time t, is

$$\mathrm{E}_t \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} U_C(C_T) \left(\frac{\tilde{p}_t(h)}{P_{H,T}} \right)^{-\sigma} Y_{H,T} \left[\frac{\tilde{p}_t(h)}{P_{H,T}} \frac{P_{H,T}}{P_T} - \mu_T \frac{V_y(\tilde{y}_{t,T}(h), \xi_T)}{U_C(C_T)} \right] \right\} = 0, \quad (17)$$

where

$$\tilde{y}_{t,T}(h) = \left(\frac{\tilde{p}_t(h)}{P_{H,T}} \right)^{-\sigma} Y_{H,T}$$

and

$$P_{H,t}^{1-\sigma} = \alpha P_{H,t-1}^{1-\sigma} + (1-\alpha) \tilde{p}_t^{1-\sigma}(h), \quad (18)$$

is the law of motion of the producer price index. Following Benigno and Woodford (2003), we take a second-order approximation to equation (17) combined with a second-order approximation to equation (18). We integrate the resulting equation forward starting from period 0 and obtain

$$
\begin{aligned}
& V_0 = \mathrm{E}_0\{\sum_{t=0}^{\infty}\beta^t[\eta\hat{Y}_{H,t} + \rho\hat{C}_t - \hat{p}_{H,t} + \hat{\mu}_t - \eta\hat{a}_t] + \\
& \frac{1}{2}[\eta\hat{Y}_{H,t} + \rho\hat{C}_t - \hat{p}_{H,t} + \hat{\mu}_t - \eta\hat{a}_t]\cdot[-\rho\hat{C}_t + (2+\eta)\hat{Y}_{H,t} \\
& +\hat{p}_{H,t} + \hat{\mu}_t - \eta\hat{a}_t] + \frac{\sigma(1+\eta)}{2k}\pi^2_{H,t}\} + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3),
\end{aligned}
\tag{19}
$$

where V_t is defined by

$$
V_t \equiv k^{-1}[\pi_{H,t} + v_\pi \pi^2_{H,t} + v_z \pi_{H,t} Z_t]
$$

with

$$
v_\pi \equiv \sigma(1+\eta) - \frac{1-\sigma}{1-\alpha}, \qquad v_z \equiv \frac{1-\alpha\beta}{2},
$$

and

$$
Z_t = [-\rho(\hat{C}_t - \hat{a}_t) + (2+\eta)\hat{Y}_{H,t} + \hat{p}_{H,t} + \hat{\mu}_t - \eta\hat{a}_t] + v_k E_t \pi_{H,t+1} + \alpha\beta E_t Z_{t+1},
$$

where

$$
v_k \equiv -\frac{\alpha\beta}{1-\alpha\beta}(1 - 2\sigma - \eta\sigma).
$$

We can write equation (19) in a vector-matrix notation as

$$
\begin{aligned}
V_0 \;=\; & E_0\{\sum_{t=0}^{\infty}\beta^t[a'_x x_t + a'_\xi \xi_t + \frac{1}{2}x'_t A_x x_t + x'_t A_\xi \xi_t + \frac{1}{2}a_{\pi_h}\pi^2_{H,t}] \\
& +\text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3),
\end{aligned}
\tag{20}
$$

where s.o.t.i.p. denotes second-order terms independent of policy and

$$
a'_x \equiv [\;\eta \;\; \rho \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0],
$$

$$
a'_\xi \equiv [-\eta \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0],
$$

$$
A_x \equiv \begin{bmatrix}
\eta(2+\eta) & \rho & -1 & 0 & 0 & 0 & 0 \\
\rho & -\rho^2 & \rho & 0 & 0 & 0 & 0 \\
-1 & \rho & -1 & 0 & 0 & 0 & 0 \\
0 & & 0 & 0 & 0 & 0 & 0 \\
0 & & 0 & 0 & 0 & 0 & 0 \\
0 & & 0 & 0 & 0 & 0 & 0 \\
0 & & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$A_\xi \equiv \begin{bmatrix} -\eta(1+\eta) & (1+\eta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$a_{\pi_h} \equiv \sigma(1+\eta)k^{-1}.$$

Repeating the same steps for the foreign country, we can obtain the second-order approximation to country F's AS equation as

$$\begin{aligned} V_0^* &= E_0\{\sum_{t=0}^{\infty}\beta^t[b_x' x_t + b_\xi' \xi_t + \frac{1}{2}x_t' B_x x_t + x_t' B_\xi \xi_t + \frac{1}{2}b_{\pi_f}\pi_{F,t}^{*2}] \\ &\quad + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{21}$$

where

$$b_x' \equiv [0 \;\; 0 \;\; 0 \;\; \eta \;\; \rho \;\; -1 \;\; 0],$$

$$b_\xi' \equiv [0 \;\; 0 \;\; 0 \;\; -\eta \;\; 1 \;\; 0],$$

$$B_x \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta(2+\eta) & \rho & -1 & 0 \\ 0 & 0 & 0 & \rho & -\rho^2 & \rho & 0 \\ 0 & 0 & 0 & -1 & \rho & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_\xi \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\eta(1+\eta) & (1+\eta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$b_{\pi_f} \equiv \sigma(1+\eta)\,(k^*)^{-1},$$

where V_t^* is defined by

$$V_t^* \equiv k^{*-1}[\pi_{F,t}^* + v_\pi \pi_{F,t}^{*2} + v_z \pi_{F,t}^* Z_t^*],$$

with

$$Z_t^* = [-\rho(\hat{C}_t^* - \hat{a}_t^*) + (2+\eta)\hat{Y}_{F,t}^* + \hat{p}_{F,t} + \hat{\mu}_t^* - \eta\hat{a}_t^*] + v_k E_t \pi_{F,t+1}^* + \alpha\beta E_t Z_{t+1}^*.$$

We now take a second-order expansion of the demand equation for the goods produced in the home country, which is the LHS equation of (2.8) in the main text

$$Y_{H,t} = \left(\frac{P_{H,t}}{P_t}\right)^{-\theta} [nC_t + (1-n)C_t^*] + G_t,$$

obtaining

$$\begin{aligned}
\hat{Y}_{H,t} \quad = \quad & -\theta s_c \hat{p}_{H,t} + n s_c \hat{C}_t + (1-n)s_c\hat{C}_t^* + \hat{G}_t + \frac{s_c n(1-ns_c)}{2}(\hat{C}_t)^2 + \\
& + \frac{s_c(1-n)(1-(1-n)s_c)}{2}(\hat{C}_t^*)^2 - n(1-n)s_c^2\hat{C}_t^*\hat{C}_t + \\
& + \frac{s_c(1-s_c)}{2}\theta^2\hat{p}_{H,t}^2 - s_c(1-s_c)\theta n\hat{C}_t\hat{p}_{H,t} - s_c(1-s_c)\theta(1-n)\hat{C}_t^*\hat{p}_{H,t} - \\
& - s_c(n\hat{C}_t + (1-n)\hat{C}_t^*)\hat{G}_t + s_c\theta\hat{p}_{H,t}\hat{G}_t + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3), \qquad (22)
\end{aligned}$$

where $\hat{G} \equiv (G_t - \bar{G})/\bar{Y}$. Equation (22) can be rewritten in a vector-matrix form as

$$\sum_{t=0}^{\infty} \beta^t [d_x' x_t + d_\xi' \xi_t + \frac{1}{2} x_t' D_x x_t + x_t' D_\xi \xi_t] + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3) = 0, \qquad (23)$$

where

$$d_x' \equiv [-1 \quad ns_c \quad -\theta s_c \quad 0 \quad (1-n)s_c \quad 0 \quad 0],$$

$$d_\xi' \equiv [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$D_x \equiv \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & ns_c(1-ns_c) & -\theta s_c(1-s_c)n & 0 & -n(1-n)s_c^2 & 0 & 0 \\
0 & -\theta s_c(1-s_c)n & s_c(1-s_c)\theta^2 & 0 & -\theta s_c(1-s_c)(1-n) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -n(1-n)s_c^2 & -\theta s_c(1-s_c)(1-n) & 0 & s_c(1-n)(1-(1-n)s_c) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$D_\xi \equiv \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -ns_c & 0 & 0 & 0 \\
0 & 0 & \theta s_c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -(1-n)s_c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

As well, we take a second-order expansion of the demand equation for the goods produced in the foreign country, which is RHS equation of (2.11) in the main text

$$Y^*_{F,t} = \left(\frac{P_{F,t}}{P_t}\right)^{-\theta}[nC_t + (1-n)C^*_t] + G^*_t$$

obtaining

$$\begin{aligned}\hat{Y}^*_{F,t} &= -\theta s_c \hat{p}_{F,t} + ns_c\hat{C}_t + (1-n)s_c\hat{C}^*_t + \hat{G}^*_t + \frac{s_c n(1-ns_c)}{2}(\hat{C}_t)^2 + \\ &+\frac{s_c(1-n)(1-(1-n)s_c)}{2}(\hat{C}^*_t)^2 - n(1-n)s_c^2\hat{C}^*_t\hat{C}_t + \\ &+\frac{s_c(1-s_c)}{2}\theta^2\hat{p}^2_{F,t} - s_c(1-s_c)\theta n\hat{C}_t\hat{p}_{F,t} - s_c(1-s_c)\theta(1-n)\hat{C}^*_t\hat{p}_{F,t} - \\ &-s_c(n\hat{C}_t + (1-n)\hat{C}^*_t)\hat{G}^*_t + s_c\theta\hat{p}_{F,t}\hat{G}^*_t + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{24}$$

where $\hat{G}^* \equiv (G^*_t - \bar{G})/\bar{Y}$. Equation (24) can be rewritten in a vector-matrix form as

$$\sum_{t=0}^{\infty}\beta^t[f'_x x_t + f'_\xi\xi_t + \frac{1}{2}x'_t F_x x_t + x'_t F_\xi \xi_t] + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3) = 0, \tag{25}$$

where

$$f'_x \equiv [0 \quad ns_c \quad 0 \quad -1 \quad (1-n)s_c \quad -\theta s_c \quad 0],$$
$$f'_\xi \equiv [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1],$$

$$F_x \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & ns_c(1-ns_c) & 0 & 0 & -n(1-n)s_c^2 & -\theta s_c(1-s_c)n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -n(1-n)s_c^2 & 0 & 0 & s_c(1-n)(1-(1-n)s_c) & -\theta s_c(1-s_c)(1-n) & 0 \\ 0 & -\theta s_c(1-s_c)n & 0 & 0 & -\theta s_c(1-s_c)(1-n) & s_c(1-s_c)\theta^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F_\xi \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -ns_c \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1-n)s_c \\ 0 & 0 & 0 & 0 & 0 & \theta s_c \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now derive the relation between relative prices and terms of trade exploiting the definition of the price index (1.1) in the main text and the fact that $T \equiv P_F/P_H$

$$\left(\frac{P_{H,t}}{P_t}\right)^{\theta-1} = n + (1-n)T_t^{1-\theta}.$$

We obtain that

$$\hat{p}_{H,t} = -(1-n)\hat{T}_t - \frac{1}{2}n(1-n)(1-\theta)\hat{T}_t^2 + \mathcal{O}(||\xi||^3), \tag{26}$$

$$\sum_{t=0}^{\infty} \beta^t [h_x' x_t + \frac{1}{2} x_t' H_x x_t] + \mathcal{O}(||\xi||^3) = 0, \tag{27}$$

where

$$h_x' \equiv [0 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; -(1-n)],$$

$$H_x \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -n(1-n)(1-\theta) \end{bmatrix}.$$

Again starting from (1.1) in the main text, we can take a second-order approximation to

$$\left(\frac{P_{F,t}}{P_t}\right)^{\theta-1} = nT_t^{\theta-1} + (1-n)$$

obtaining

$$\hat{p}_{F,t} = n\hat{T}_t - \frac{1}{2}n(1-n)(1-\theta)\hat{T}_t^2 + \mathcal{O}(||\xi||^3), \tag{28}$$

$$\sum_{t=0}^{\infty} \beta^t [l_x' x_t + \frac{1}{2} x_t' L_x x_t] + \mathcal{O}(||\xi||^3) = 0, \tag{29}$$

where

$$l_x{}' \equiv [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; n],$$

$$L_x \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -n(1-n)(1-\theta) \end{bmatrix}.$$

The final equation that we need to consider is the risk-sharing condition, equation (1.3) in the main text

$$U_C(C_t) = U_C(C_t^*)$$

which is exactly log-linear with isoelastic preferences

$$\hat{C}_t = \hat{C}_t^*, \tag{30}$$

and can be written as

$$\sum_{t=0}^{\infty} \beta^t [m_x' x_t] = 0, \tag{31}$$

where

$$m_x' \equiv [0 \;\; 1 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0],$$

We now proceed to construct a quadratic approximation to the welfare criteria for the home and foreign countries. To this purpose, we combine constraints (20), (21), (23), (25), (27), (29), (31) to get rid of the linear terms in the expansions (14) and (16). In particular we need to take a particular linear combination of those constraints. We collect the vectors that multiply the endogenous variables in the linear components of the above constraints in the following (7×7) matrix

$$\Gamma \equiv [a_x \; b_x \; d_x \; f_x \; h_x \; l_x \; m_x].$$

In order to get the right weights to eliminate the linear terms in the welfare approximation of the Home country, equation (14), we solve the following system of linear equations for the vector ζ

$$\Gamma\zeta = z_x.$$

As well, we solve the system

$$\Gamma\zeta^* = z_x^*$$

to eliminate the linear terms in the second-order approximation to country's F welfare, equation (16). We obtain

$$\begin{aligned} W \;\; = \;\; & -\frac{1}{2}\bar{U}_C\bar{C}E_0\{\sum_{t=0}^{\infty}\beta^t[x_t'Q_x x_t + 2x_t'Q_\xi \xi_t + q_{\pi_h}\pi_{H,t}^2 + q_{\pi_f}\pi_{F,t}^{*2}]\} + \\ & + K_0 + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{32}$$

where

$$Q_x = Z_x + \zeta_1 A_x + \zeta_2 B_x + \zeta_3 D_x + \zeta_4 F_x,$$
$$Q_\xi = Z_\xi + \zeta_1 A_\xi + \zeta_2 B_\xi + \zeta_3 D_\xi + \zeta_4 F_\xi,$$
$$q_{\pi_h} = z_{\pi_h} + \zeta_1 a_{\pi_h},$$
$$q_{\pi_f} = \zeta_2 b_{\pi_f},$$

and K_0 is defined as

$$K_0 \equiv \bar{U}_C\bar{C}[\zeta_1 V_0 + \zeta_2 V_0^*].$$

For country F we obtain that

$$
\begin{aligned}
W^* &= -\frac{1}{2}\bar{U}_C\bar{C}E_0\{\sum_{t=0}^{\infty}\beta^t[x_t'Q_x^*x_t + 2x_t'Q_\xi^*\xi_t + q_{\pi_h}^*\pi_{H,t}^2 + q_{\pi_f}^*\pi_{F,t}^{*2}]\} + \\
&\quad + K_0^* + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \qquad (33)
\end{aligned}
$$

where

$$
\begin{gathered}
Q_x^* = Z_x^* + \zeta_1^*A_x + \zeta_2^*B_x + \zeta_3^*D_x + \zeta_4^*F_x, \\
Q_\xi^* = Z_\xi^* + \zeta_1^*A_\xi + \zeta_2^*B_\xi + \zeta_3^*D_\xi + \zeta_4^*F_\xi, \\
q_{\pi_h}^* = \zeta_1^*a_{\pi_h}, \\
q_{\pi_f}^* = z_{\pi_f}^* + \zeta_2^*b_{\pi_f}, \\
K_0^* = \bar{U}_C\bar{C}[\zeta_1^*V_0 + \zeta_2^*V_0^*].
\end{gathered}
$$

We further note that by using the set of structural equilibrium conditions (22), (24), (26), (28) and (30) up to first-order terms we can write

$$
x_t = N_x y_t + N_\xi \xi_t + \mathcal{O}(||\xi||^2), \qquad (34)
$$

where $y_t' = [\hat{C}_t \;\; \hat{T}_t]$ and

$$
N_x = \begin{bmatrix} s_c & s_c\theta(1-n) \\ 1 & 0 \\ 0 & -(1-n) \\ s_c & -s_c\theta n \\ 1 & 0 \\ 0 & n \\ 0 & 1 \end{bmatrix},
$$

$$
N_\xi = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

By substituting (34) into (32) and (33), we obtain that

$$
\begin{aligned}
W &= -\frac{1}{2}\bar{U}_C\bar{C}E_0\{\sum_{t=0}^{\infty}\beta^t[y_t'\tilde{Q}_x y_t + 2y_t'\tilde{Q}_\xi\xi_t + q_{\pi_h}\pi_{H,t}^2 + q_{\pi_f}\pi_{F,t}^{*2}]\} \\
&\quad + K_0 + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \qquad (35)
\end{aligned}
$$

$$W^* = -\frac{1}{2}\bar{U}_C\bar{C}E_0\{\sum_{t=0}^{\infty}\beta^t[y_t'\tilde{Q}_x^* y_t + 2y_t'\tilde{Q}_\xi^*\xi_t + q_{\pi_h}^*\pi_{H,t}^2 + q_{\pi_f}^*\pi_{F,t}^{*2}]\} + K_0^* + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \quad (36)$$

where

$$\tilde{Q}_x = N_x' Q_x N_x,$$
$$\tilde{Q}_\xi = N_x' Q_x N_\xi + N_x' Q_\xi,$$
$$\tilde{Q}_x^* = N_x' Q_x^* N_x,$$
$$\tilde{Q}_\xi^* = N_x' Q_x^* N_\xi + N_x' Q_\xi^*.$$

Starting from equations (35) and (36) which are written using matrix notation, we build a more transparent quadratic form in terms of target variables. To this end, we note that the world welfare

$$W^W = nW + (1-n)W^*,$$

which is defined as the weighted average of country H's and F's welfares with weigths n and $(1-n)$, can be written as

$$W^W = -\frac{1}{2}\bar{U}_C\bar{C}E_0\{\sum_{t=0}^{\infty}\beta^t[y_t'\tilde{Q}_x^W y_t + 2y_t'\tilde{Q}_\xi^W\xi_t + q_{\pi_h}^W\pi_{H,t}^2 + q_{\pi_f}^W\pi_{F,t}^{*2}] + K_0^W + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \quad (37)$$

where the elements of the matrices are the followings

$$\tilde{Q}_{x,11}^W = \frac{(s_c\eta+\rho)^2 + \bar{\mu}^{-1}(\bar{\mu}-1)(s_c-\rho)(s_c\eta+\rho) - \bar{\mu}^{-1}(\bar{\mu}-1)\rho(1-s_c)}{s_c\eta+\rho},$$

$$\tilde{Q}_{x,12}^W = 0,$$
$$\tilde{Q}_{x,21}^W = 0,$$

$$\tilde{Q}_{x,22}^W = (1-n)n\theta\frac{(s_c\eta+\rho)(1+\eta s_c\theta) + \bar{\mu}^{-1}(\bar{\mu}-1)(s_c-\rho)(1+\eta s_c\theta) - \bar{\mu}^{-1}(\bar{\mu}-1)(1-s_c)}{s_c\eta+\rho},$$

and

$$\tilde{Q}_{\xi,11}^W = n\frac{\eta}{s_c\eta+\rho}[\bar{\mu}^{-1}(\bar{\mu}-1)(\rho-s_c) - (s_c\eta+\rho)],$$

$$\tilde{Q}_{\xi,12}^W = n\frac{\bar{\mu}^{-1}(\bar{\mu}-1)s_c(1+\eta)}{s_c\eta+\rho},$$

$$\tilde{Q}_{\xi,13}^W = n\frac{1}{s_c\eta+\rho}[\bar{\mu}^{-1}(\bar{\mu}-1)(s_c\eta+\rho) - \bar{\mu}^{-1}(\bar{\mu}-1)\eta\rho + \eta(s_c\eta+\rho)],$$

$$\tilde{Q}^W_{\xi,14} = n^{-1}(1-n)\tilde{Q}^W_{\xi,11},$$
$$\tilde{Q}^W_{\xi,15} = n^{-1}(1-n)\tilde{Q}^W_{\xi,13},$$
$$\tilde{Q}^W_{\xi,16} = n^{-1}(1-n)\tilde{Q}^W_{\xi,14},$$
$$\tilde{Q}^W_{\xi,21} = (1-n)\theta\tilde{Q}^W_{\xi,11},$$
$$\tilde{Q}^W_{\xi,22} = (1-n)\theta\tilde{Q}^W_{\xi,12},$$
$$\tilde{Q}^W_{\xi,23} = (1-n)\theta\tilde{Q}^W_{\xi,13} + n(1-n)\bar{\mu}^{-1}(\bar{\mu}-1)\frac{1-\rho\theta}{(s_c\eta+\rho)},$$
$$\tilde{Q}^W_{\xi,24} = -\tilde{Q}^W_{\xi,21},$$
$$\tilde{Q}^W_{\xi,25} = -\tilde{Q}^W_{\xi,22},$$
$$\tilde{Q}^W_{\xi,26} = -\tilde{Q}^W_{\xi,23},$$
$$q^W_{\pi_h} = \left(\bar{\mu}^{-1}(\bar{\mu}-1)s_c + s_c\eta + \rho - \bar{\mu}^{-1}(\bar{\mu}-1)\rho\right)n\frac{\sigma}{s_c k(s_c\eta+\rho)},$$
$$q^W_{\pi_f} = \left(\bar{\mu}^{-1}(\bar{\mu}-1)s_c + s_c\eta + \rho - \bar{\mu}^{-1}(\bar{\mu}-1)\rho\right)(1-n)\frac{\sigma}{s_c k^*(s_c\eta+\rho)},$$
$$K^W_0 \equiv nK_0 + (1-n)K^*_0.$$

We observe that we can write (37) in the form

$$W^W = -\frac{1}{2}\bar{U}_C\bar{C}E_0\{\sum_{t=0}^{\infty}\beta^t L^W_t\} + K^W_0 + \text{t.i.p.} + \mathcal{O}(||\xi||^3),$$

where

$$L^W_t = \lambda^w_c(\hat{C}_t - \tilde{C}^w_t)^2 + \tilde{\lambda}^w_q(\tilde{T}_t - \tilde{T}^w_t)^2 + n\lambda^w_{\pi_h}\pi^2_{H,t} + (1-n)\lambda^w_{\pi_f}\pi^{*2}_{F,t}, \tag{38}$$

$$\lambda^w_c \equiv \tilde{Q}^W_{x,11}, \qquad \tilde{\lambda}^w_q \equiv \tilde{Q}^W_{x,22},$$
$$\lambda^w_{\pi_h} = n^{-1}q^W_{\pi_h}, \qquad \lambda^w_{\pi_f} = (1-n)^{-1}q^W_{\pi_f},$$

where $\tilde{C}^w_{W,t}$ is defined as

$$\tilde{C}^w_{W,t} = -(n\lambda^w_c)^{-1}[\tilde{Q}^W_{\xi,11}\hat{a}_{W,t} + \tilde{Q}^W_{\xi,12}\hat{\mu}_{W,t} + \tilde{Q}^W_{\xi,13}\hat{G}_{W,t}], \tag{39}$$

and $\tilde{T}^w_t$ as

$$\tilde{T}^w_t = -(\tilde{\lambda}^w_q)^{-1}[\tilde{Q}^W_{\xi,21}\hat{a}_{R,t} + \tilde{Q}^W_{\xi,22}\hat{\mu}_{R,t} + \tilde{Q}^W_{\xi,23}\hat{G}_{R,t}], \tag{40}$$

where $\hat{a}_{W,t} \equiv n\hat{a}_t + (1-n)\hat{a}^*_t$, and $\hat{a}_{R,t} \equiv \hat{a}_t - \hat{a}^*_t$. (The same definitions apply to the other shocks)

We further note that

$$\tilde{\lambda}_q^w = (1-n)n\theta[z_1\lambda_c^w + z_2],$$

where

$$z_1 \equiv \frac{(1+\theta s_c\eta)}{s_c\eta+\rho} \qquad\qquad z_2 \equiv \frac{\bar{\mu}^{-1}(\bar{\mu}-1)(1-s_c)s_c\eta(\rho\theta-1)}{(s_c\eta+\rho)^2}$$

with the consequence that when $\rho\theta = 1$, then $z_2 = 0$ and $z_1 = \theta$.

We can now write (38) in another form, noting that

$$\tilde{Y}_{H,t}^w \equiv s_c\tilde{C}_t^w + (1-n)\theta s_c\tilde{T}_t^w + \hat{G}_t,$$

$$\tilde{Y}_{F,t}^w \equiv s_c\tilde{C}_t^w - n\theta s_c\tilde{T}_t^w + \hat{G}_t^*,$$

and that

$$(\hat{C}_t - \tilde{C}_t^w)^2 = ns_c^{-2}(\hat{Y}_{H,t} - \tilde{Y}_{H,t}^w)^2 + (1-n)s_c^{-2}(\hat{Y}_{F,t}^* - \tilde{Y}_{F,t}^w)^2 - n(1-n)\theta^2(\tilde{T}_t - \tilde{T}_t^w)^2.$$

From the above conditions, it follows that

$$\begin{aligned} L_t^W = {} & n\lambda_y^w(\hat{Y}_{H,t} - \tilde{Y}_{H,t}^w)^2 + (1-n)\lambda_y^w(\hat{Y}_{F,t}^* - \tilde{Y}_{F,t}^w)^2 + \\ & n(1-n)\lambda_q^w(\tilde{T}_t - \tilde{T}_t^w)^2 + n\lambda_{\pi_h}^w\pi_{H,t}^2 + (1-n)\lambda_{\pi_f}^w\pi_{F,t}^{*2} \end{aligned} \tag{41}$$

where now

$$\lambda_y^w \equiv s_c^{-2}\lambda_c^w,$$

$$\lambda_q^w \equiv \frac{\theta(1-\theta\rho)}{(s_c\eta+\rho)}[(s_c\eta+\rho) + \bar{\mu}^{-1}(\bar{\mu}-1)(s_c-1) + \bar{\mu}^{-1}(\bar{\mu}-1)(s_c-\rho)].$$

In particular we have defined

$$\tilde{Y}_{H,t}^w \equiv c_1\hat{a}_t + c_2\hat{a}_t^* + c_3\hat{\mu}_t + c_4^*\hat{\mu}_t^* + c_5\hat{G}_t + c_6\hat{G}_t^*,$$

$$\tilde{Y}_{F,t}^w \equiv d_1\hat{a}_t + d_2\hat{a}_t^* + d_3\hat{\mu}_t + d_4\hat{\mu}_t^* + d_5\hat{G}_t + d_6\hat{G}_t^*,$$

$$c_1 \equiv -\frac{s_c Q_{\xi,11}^W[\lambda_c^w(nz_1 + (1-n)\theta) + nz_2]}{n\lambda_c^w(z_1\lambda_c^w + z_2)},$$

$$c_2 \equiv -\frac{s_c\tilde{Q}_{\xi,11}^W(1-n)[\lambda_c^w(z_1-\theta) + z_2]}{n\lambda_c^w(z_1\lambda_c^w + z_2)},$$

$$c_3 \equiv -\frac{s_c\tilde{Q}_{\xi,12}^W[\lambda_c^w(nz_1 + (1-n)\theta) + nz_2]}{n\lambda_c^w(z_1\lambda_c^w + z_2)},$$

$$c_4 \equiv -\frac{s_c \tilde{Q}^W_{\xi,12}(1-n)\left[\lambda^w_c(z_1-\theta)+z_2\right]}{n\lambda^w_c(z_1\lambda^w_c+z_2)},$$

$$c_5 \equiv -\left[\frac{s_c \tilde{Q}^W_{\xi,13}\lambda^w_c(nz_1+(1-n)\theta)+s_c \tilde{Q}^W_{\xi,13} n z_2 + n(1-n)s_c\lambda^w_c \bar{\mu}^{-1}\left(\bar{\mu}-1\right)\frac{1-\rho\theta}{(s_c\eta+\rho)}}{n\lambda^w_c(z_1\lambda^w_c+z_2)}\right]+1,$$

$$c_6 \equiv -\left[\frac{s_c \tilde{Q}^W_{\xi,13}\lambda^w_c(1-n)(z_1-\theta)+s_c \tilde{Q}^W_{\xi,13}(1-n)z_2 - n(1-n)s_c\lambda^w_c \bar{\mu}^{-1}\left(\bar{\mu}-1\right)\frac{1-\rho\theta}{(s_c\eta+\rho)}}{n\lambda^w_c(z_1\lambda^w_c+z_2)}\right],$$

$$d_1 \equiv -\frac{s_c \tilde{Q}^W_{\xi,11}\left[\lambda^w_c(z_1-\theta)+z_2\right]}{\lambda^w_c(z_1\lambda^w_c+z_2)},$$

$$d_2 \equiv -\frac{s_c \tilde{Q}^W_{\xi,11}\left[\lambda^w_c[(1-n)z_1+n\theta]+(1-n)z_2\right]}{n\lambda^w_c(z_1\lambda^w_c+z_2)},$$

$$d_3 \equiv -\frac{s_c \tilde{Q}^W_{\xi,12}\left[\lambda^w_c(z_1-\theta)+z_2\right]}{\lambda^w_c(z_1\lambda^w_c+z_2)},$$

$$d_4 \equiv -\frac{s_c \tilde{Q}^W_{\xi,12}\left[\lambda^w_c[(1-n)z_1+n\theta]+(1-n)z_2\right]}{n\lambda^w_c(z_1\lambda^w_c+z_2)},$$

$$d_5 \equiv -\left[\frac{s_c \tilde{Q}^W_{\xi,13}\lambda^w_c(z_1-\theta)+s_c \tilde{Q}^W_{\xi,13}z_2 - n s_c\lambda^w_c \bar{\mu}^{-1}\left(\bar{\mu}-1\right)\frac{1-\rho\theta}{(s_c\eta+\rho)}}{\lambda^w_c(z_1\lambda^w_c+z_2)}\right],$$

$$d_6 \equiv -\left[\frac{s_c \tilde{Q}^W_{\xi,13}\lambda^w_c((1-n)z_1+n\theta)+s_c \tilde{Q}^W_{\xi,13}(1-n)z_2 + n^2 s_c\lambda^w_c \bar{\mu}^{-1}\left(\bar{\mu}-1\right)\frac{1-\rho\theta}{(s_c\eta+\rho)}}{n\lambda^w_c(z_1\lambda^w_c+z_2)}\right]+1.$$

We can further use (35) and (36) to get the relative welfare criterion

$$W^R = W - W^*$$

defined as the difference between the country H's and F's welfare criteria. We obtain

$$W^R = -\frac{1}{2}\bar{U}_C\bar{C}E_0\left\{\sum_{t=0}^{\infty}\beta^t L^R_t\right\} + K^R_0 + \text{t.i.p.} + \mathcal{O}(||\xi||^3),$$

where

$$L^R_t \equiv y_t'\tilde{Q}^R_x y_t + 2y_t'\tilde{Q}^R_\xi \xi_t + q^R_{\pi_h}\pi^2_{H,t} + q^R_{\pi_f}\pi^{*2}_{F,t}, \tag{42}$$

and the elements of the matrices are

$$\tilde{Q}^R_{x,11} = 0,$$

$$\tilde{Q}^R_{x,12} = \frac{\bar{\mu}^{-1}\theta}{(1+\eta\theta s_c)}[(1-s_c) + (s_c\eta + \rho)(1-\theta s_c)],$$

$$\tilde{Q}^R_{x,21} = \tilde{Q}^R_{x,12},$$

$$\tilde{Q}^R_{x,22} = \frac{(1-2n)\bar{\mu}^{-1}\theta}{(1+\eta\theta s_c)}[(1+\eta\theta s_c)(1-\theta s_c) + \theta(1-s_c)],$$

$$\tilde{Q}^R_{\xi,11} = -\frac{\eta}{(1+\eta\theta s_c)}\bar{\mu}^{-1}(1-\theta s_c),$$

$$\tilde{Q}^R_{\xi,12} = -\bar{\mu}^{-1}\frac{s_c\theta(1+\eta)}{(1+\eta\theta s_c)},$$

$$\tilde{Q}^R_{\xi,13} = -\frac{\rho\theta + \eta\theta s_c - \eta}{(1+\eta\theta s_c)}\bar{\mu}^{-1},$$

$$\tilde{Q}^R_{\xi,14} = -\tilde{Q}^R_{\xi,11},$$

$$\tilde{Q}^R_{\xi,15} = -\tilde{Q}^R_{\xi,12},$$

$$\tilde{Q}^R_{\xi,16} = -\tilde{Q}^R_{\xi,13},$$

$$\tilde{Q}^R_{\xi,21} = (1-n)\theta\tilde{Q}^R_{\xi,11},$$

$$\tilde{Q}^R_{\xi,22} = (1-n)\theta\tilde{Q}^R_{\xi,12},$$

$$\tilde{Q}^C_{\xi,23} = -(1-n)\frac{1+\eta\theta s_c - \eta}{(1+\eta\theta s_c)}\bar{\mu}^{-1},$$

$$\tilde{Q}^R_{\xi,24} = n(1-n)^{-1}\tilde{Q}^R_{\xi,21},$$

$$\tilde{Q}^R_{\xi,25} = n(1-n)^{-1}\tilde{Q}^R_{\xi,22},$$

$$\tilde{Q}^R_{\xi,26} = n(1-n)^{-1}\tilde{Q}^R_{\xi,23},$$

$$q^R_{\pi_h} = \bar{\mu}^{-1}(1-\theta s_c)\frac{\sigma}{s_c k(1+\eta\theta s_c)},$$

$$q^R_{\pi_f} = -\bar{\mu}^{-1}(1-\theta s_c)\frac{\sigma}{s_c k^*(1+\eta\theta s_c)},$$

$$K^R_0 = K_0 - K^*_0.$$

We can further write

$$L^R_t = 2\lambda^R_{yq}(\hat{C}_t - \tilde{C}^R_t)(\hat{T}_t - \tilde{T}^{r_1}_t) + \lambda^R_q(\hat{T}_t - \tilde{T}^{r_2}_t)^2 + \lambda^R_{\pi_h}\pi^2_{H,t} + \lambda^R_{\pi_f}\pi^{*2}_{F,t} \qquad (43)$$

where

$$\lambda^R_{yq} \equiv \tilde{Q}^R_{x,12},$$

$$\lambda^R_q \equiv \tilde{Q}^R_{x,22},$$

$$\lambda^R_{\pi_h} \equiv q^R_{\pi_h} \qquad \lambda^R_{\pi_f} \equiv q^R_{\pi_f},$$

and $\tilde{C}^R_t$ is defined by

$$\tilde{C}^R_t \equiv -(\lambda^R_{yq})^{-1}[b_1\hat{a}_{W,t} + b_2\hat{u}_{W,t} + b_3\hat{G}_{W,t}], \tag{44}$$

with

$$b_1 \equiv (1-n)^{-1}\tilde{Q}^R_{\xi,21},$$

$$b_2 \equiv (1-n)^{-1}\tilde{Q}^R_{\xi,22}, \qquad b_3 \equiv (1-n)^{-1}\tilde{Q}^R_{\xi,23}.$$

$\tilde{T}^{r_1}_t$ is defined as

$$\tilde{T}^{r_1}_t \equiv -(\lambda^R_{yq})^{-1}[b_4\hat{a}_{R,t} + b_5\hat{\mu}_{R,t} + b_6\hat{G}_{R,t}], \tag{45}$$

$$b_4 \equiv b_1\theta^{-1},$$

$$b_5 \equiv b_2\theta^{-1}, \qquad b_6 \equiv b_3\theta^{-1} + \frac{\bar{\mu}^{-1}(1-\rho\theta)}{(1+\eta\theta s_c)},$$

and finally

$$\tilde{T}^{r_2}_t \equiv -(1-2n)(\lambda^R_q)^{-1}[b_1\hat{a}_{R,t} + b_2\hat{\mu}_{R,t} + b_3\hat{G}_{R,t}].$$

As another way to write the relative welfare criterion we can define

$$\tilde{Y}^R_{H,t} \equiv s_c\tilde{C}^R_t + (1-n)\theta s_c\tilde{T}^{r_1}_t + \hat{G}_t,$$

$$\tilde{Y}^R_{F,t} \equiv s_c\tilde{C}^R_t - n\theta s_c\tilde{T}^{r_1}_t + \hat{G}^*_t,$$

and observe that

$$2\theta(\hat{C}_t - \tilde{C}^R_t)(\hat{T}_t - \tilde{T}^{r_1}_t) = s_c^{-2}(\hat{Y}_{H,t} - \tilde{Y}^R_{H,t})^2 - s_c^{-2}(\hat{Y}^*_{F,t} - \tilde{Y}^R_{F,t})^2 - (1-2n)\theta^2(\hat{T}_t - \tilde{T}^{r_1}_t)^2. \tag{46}$$

By using (46) into (43), we obtain

$$L^R_t = \lambda^R_{y_h}(\hat{Y}_{H,t} - \tilde{Y}^R_{H,t})^2 - \lambda^R_{y_f}(\hat{Y}^*_{F,t} - \tilde{Y}^R_{F,t})^2 + \tilde{\lambda}^R_q(\hat{T}_t - \tilde{T}^R_t)^2 + \lambda^R_{\pi_h}\pi^2_{H,t} + \lambda^R_{\pi_f}\pi^2_{F,t}, \tag{47}$$

where

$$\lambda^R_{y_h} = \lambda^R_{y_f} = s_c^{-2}\lambda^R_{yq}/\theta,$$

$$\tilde{\lambda}^R_q \equiv \lambda^R_q - \theta(1-2n)\lambda^R_{yq} = \frac{\theta(1-2n)\bar{\mu}^{-1}(1-\theta s_c)(1-\theta\rho)}{(1+\eta\theta s_c)},$$

$$\tilde{T}^R_t \equiv (\tilde{\lambda}^R_q)^{-1}[\lambda^R_q\tilde{T}^{r_2}_t - \theta(1-2n)\lambda^R_{yq}\tilde{T}^{r_1}_t].$$

Now we note that

$$\tilde{Y}^R_{H,t} \equiv -(\lambda^R_{yq})^{-1}[h_1\hat{a}_t + h_3\hat{\mu}_t + h_5\hat{G}_t + h_6\hat{G}^*_t],$$

$$\tilde{Y}^R_{F,t} \equiv -(\lambda^R_{yq})^{-1}[k_2\hat{a}^*_t + k_4\hat{\mu}^*_t + k_5\hat{G}_t + k_6\hat{G}^*_t],$$
$$h_1 \equiv s_c b_{1,}$$
$$h_3 \equiv s_c b_2,$$
$$h_5 \equiv -(\lambda^R_{yq}) + s_c b_3 + \frac{\theta(1-n)s_c\bar{\mu}^{-1}(1-\theta\rho)}{(1+\eta\theta s_c)},$$
$$h_6 \equiv -\frac{\theta(1-n)s_c\bar{\mu}^{-1}(1-\theta\rho)}{(1+\eta\theta s_c)},$$
$$k_2 \equiv s_c b_1,$$
$$k_4 \equiv s_c b_2,$$
$$k_5 \equiv -\frac{\theta n s_c\bar{\mu}^{-1}(1-\theta\rho)}{(1+\eta\theta s_c)},$$
$$k_6 \equiv -\lambda^R_{yq} + s_c b_3 + \frac{\theta n s_c\bar{\mu}^{-1}(1-\theta\rho)}{(1+\eta\theta s_c)},$$

where

$$\tilde{T}^R_t \equiv (\tilde{\lambda}^R_q)^{-1}\left[\frac{\theta(1-2n)\bar{\mu}^{-1}(1-\theta\rho)}{(1+\eta\theta s_c)}\hat{G}_{R,t}\right].$$

We are now ready to retrieve in a more transparent form the welfare of the single countries. Indeed, we note that the welfare of country H can be written as

$$W = W^W + (1-n)W^R$$

By using (41) and (47), we obtain that

$$W = -\frac{1}{2}\bar{U}_C\bar{C}E_0\{\sum_{t=0}^{\infty}\beta^t L_t\} + K_0 + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \tag{48}$$

where

$$L_t = \lambda_{y_h}(\hat{Y}_{H,t} - \tilde{Y}^h_{H,t})^2 + \lambda_{y_f}(\hat{Y}^*_{F,t} - \tilde{Y}^h_{F,t})^2 + \lambda_q(\tilde{T}_t - \tilde{T}^h_t)^2 + \lambda_{\pi_h}\pi^2_{H,t} + \lambda_{\pi_f}\pi^{*2}_{F,t}, \tag{49}$$
$$\lambda_{y_h} \equiv [n\lambda^w_y + (1-n)\lambda^R_{y_h}],$$
$$\lambda_{y_f} \equiv (1-n)(\lambda^w_y - \lambda^R_{y_f}),$$
$$\lambda_{\pi_h} \equiv (n\lambda^w_{\pi_h} + (1-n)\lambda^R_{\pi_h}),$$
$$\lambda_{\pi_f} \equiv ((1-n)\lambda^w_{\pi_f} + (1-n)\lambda^R_{\pi_f}),$$
$$\lambda_q \equiv (n(1-n)\lambda^w_q + (1-n)\tilde{\lambda}^R_q),$$

$$\tilde{Y}^h_{H,t} \equiv (\lambda_{y_h})^{-1}(n\lambda^w_y \tilde{Y}^w_{H,t} + (1-n)\lambda^R_{y_h} \tilde{Y}^R_{H,t}),$$

$$\tilde{Y}^h_{F,t} \equiv (\lambda_{y_f})^{-1}(1-n)(\lambda^w_y \tilde{Y}^w_{F,t} - \lambda^R_{y_f} \tilde{Y}^R_{F,t}),$$

$$\tilde{T}^h_t \equiv (\lambda_q)^{-1}(\lambda^w_q \tilde{T}^w_t + (1-n)\tilde{\lambda}^R_q \tilde{T}^R_t).$$

The welfare of country F can be written as

$$W^* = W^W - nW^R$$

and we obtain that

$$W^* = -\frac{1}{2}\bar{U}_C \bar{C} E_0 \{\sum_{t=0}^{\infty} \beta^t L^*_t\} + K^*_0 + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \tag{50}$$

$$L^*_t = \lambda^*_{y_h}(\hat{Y}_{H,t} - \tilde{Y}^f_{H,t})^2 + \lambda^*_{y_f}(\hat{Y}^*_{F,t} - \tilde{Y}^f_{F,t})^2 + \lambda^*_q(\tilde{T}_t - \tilde{T}^f_t)^2 + \lambda^*_{\pi_h}\pi^2_{H,t} + \lambda^*_{\pi_f}\pi^{*2}_{F,t}, \tag{51}$$

$$\lambda^*_{y_h} \equiv n(\lambda^w_{y_h} - \lambda^R_{y_h}),$$

$$\lambda^*_{y_f} \equiv [(1-n)\lambda^w_{y_f} + n\lambda^R_{y_f}],$$

$$\lambda^*_{\pi_h} \equiv (n\lambda^w_{\pi_h} - n\lambda^R_{\pi_h}),$$

$$\lambda^*_{\pi_f} \equiv ((1-n)\lambda^w_{\pi_f} - n\lambda^R_{\pi_f}),$$

$$\lambda^*_q \equiv (\lambda^w_q - n\tilde{\lambda}^R_q),$$

$$\tilde{Y}^f_{H,t} \equiv (\lambda^*_{y_h})^{-1}(n\lambda^w_{y_h} \tilde{Y}^w_{H,t} - n\lambda^R_{y_h} \tilde{Y}^R_{H,t}),$$

$$\tilde{Y}^f_{F,t} \equiv (\lambda^*_{y_f})^{-1}[(1-n)\lambda^w_{y_f} \tilde{Y}^w_{F,t} + n\lambda^R_{y_f} \tilde{Y}^R_{F,t}],$$

$$\tilde{T}^f_t \equiv (\lambda^*_q)^{-1}(\lambda^w_q \tilde{T}^w_t - n\tilde{\lambda}^R_q \tilde{T}^R_t).$$

We are assuming that policymakers are committed to additional commitments at time 0 on the variables, F_t, K_t, F^*_t and K^*_t. This assumption boils down to ask that each policymaker takes as given V_0 and V^*_0 as functions of predetermined and exogenous variables and of the strategy of the other policymaker.[1] These functions will be self consistent, reflecting the time consistency of the solution searched, meaning that they will be the same functions that will be obtained under the equilibrium at later times. By inspecting the definitions of V_0 and V^*_0 we note that indeed they depend on transitional elements that are related to the specialty of time 0. The timeless perspective assumption implies that, in the above-derived welfare functions, the terms K_0, K^*_0 can be considered as given when maximizing the welfare. It follows that an equivalent way to represent the maximization of the welfare of each country is to

[1] They can also be considered as only functions of the exogenous and predetermined variables.

minimize the discounted sum of the quadratic losses, L_t and L_t^* for policymakers H and F, respectively.

The loss functions (41), (49) and (51) correspond to equations (3.21), (3.32), (3.33) in the main text.

A linear-quadratic model (section 3 of the paper). Derivation of equations (3.22), (3.23) and (3.24).

The loss functions (49), (51) and (41) are indeed quadratic and can be correctly evaluated by a log-linear approximation to the equilibrium conditions (20), (21), (23), (25), (27), (29), (31). Up to first-order terms, these equilibrium conditions imply in the same order

$$\pi_{H,t} = k(\eta\hat{Y}_{H,t} + \rho\hat{C}_t - \hat{p}_{H,t} + \hat{\mu}_t - \eta\hat{a}_t - \rho\hat{g}_t) + \beta E_t\pi_{H,t+1} \tag{52}$$

$$\pi^*_{F,t} = k^*(\eta\hat{Y}^*_{F,t} + \rho\hat{C}^*_t - \hat{p}_{F,t} + \hat{\mu}^*_t - \eta\hat{a}^*_t - \rho\hat{g}^*_t) + \beta E_t\pi^*_{F,t+1} \tag{53}$$

$$\hat{Y}_{H,t} = -\theta s_c\hat{p}_{H,t} + ns_c\hat{C}_t + (1-n)s_c\hat{C}^*_t + \hat{G}_t \tag{54}$$

$$\hat{Y}^*_{F,t} = -\theta s_c\hat{p}_{F,t} + ns_c\hat{C}_t + (1-n)s_c\hat{C}^*_t + \hat{G}^*_t \tag{55}$$

$$\hat{p}_{H,t} = -(1-n)\hat{T}_t \tag{56}$$

$$\hat{p}_{F,t} = n\hat{T}_t \tag{57}$$

$$\hat{C}_t = \hat{C}^*_t \tag{58}$$

We can use equations (54)-(58) into (52) and (53) to obtain

$$\pi_{H,t} = k[(\eta + \rho s_c^{-1})\hat{Y}_{H,t} + (1-n)(1-\theta\rho)\hat{T}_t + \hat{\mu}_t - \eta\hat{a}_t - \rho\hat{g}_{W,t} - \rho s_c^{-1}\hat{G}_t] + \beta E_t\pi_{H,t+1},$$

$$\pi^*_{F,t} = k^*[(\eta + \rho s_c^{-1})\hat{Y}^*_{F,t} - n(1-\theta\rho)\hat{T}_t + \hat{\mu}^*_t - \eta\hat{a}^*_t - \rho\hat{g}_{W,t} - \rho s_c^{-1}\hat{G}^*_t] + \beta E_t\pi^*_{F,t+1},$$

$$\hat{T}_t = \theta^{-1}s_c^{-1}(\hat{Y}_{H,t} - \hat{Y}^*_{F,t})$$

which can be further rewritten as

$$\pi_{H,t} = \kappa[(\hat{Y}_{H,t} - \tilde{Y}^w_{H,t}) + (1-n)\psi(\hat{T}_t - \tilde{T}^w_t) + u_t] + \beta E_t\pi_{H,t+1}, \tag{59}$$

$$\pi^*_{F,t} = \kappa^*[(\hat{Y}^*_{F,t} - \tilde{Y}^w_{F,t}) - n\psi(\hat{T}_t - \tilde{T}^w_t) + u^*_t] + \beta E_t\pi^*_{F,t+1}, \tag{60}$$

$$(\hat{T}_t - \tilde{T}^w_t) = \theta^{-1}s_c^{-1}[(\hat{Y}_{H,t} - \tilde{Y}^w_{H,t}) - (\hat{Y}^*_{F,t} - \tilde{Y}^w_{F,t})], \tag{61}$$

where we have defined $\kappa^i \equiv k^i(\rho s_c^{-1} + \eta)$ and $\psi \equiv (1-\rho\theta)/(\rho s_c^{-1} + \eta)$ and

$$u_t \equiv (\rho s_c^{-1} + \eta)^{-1}[\hat{\mu}_t - \eta\hat{a}_t - \rho s_c^{-1}\hat{G}_t + (\eta + \rho s_c^{-1})\tilde{Y}^w_{H,t} + (1-n)(1-\theta\rho)\tilde{T}^w_t],$$

$$u^*_t \equiv (\rho s_c^{-1} + \eta)^{-1}[\hat{\mu}^*_t - \eta\hat{a}^*_t - \rho s_c^{-1}\hat{G}^*_t + (\eta + \rho s_c^{-1})\tilde{Y}^w_{F,t} - n(1-\theta\rho)\tilde{T}^w_t].$$

Equations (59), (60), (61) corresponds to equations (3.13), (3.14) and (3.15) in the main text. To these equations, we need to add the log-linear approximation to the constraint given by the timeless perspective equilibrium (V_0 and V_0), which up to a first-order terms requires an initial constraint on $\pi_{H,0} = \bar{\pi}_{H,0}$ and $\pi^*_{F,0} = \bar{\pi}^*_{F,0}$ where as in Woodford (2003, ch. 7), $\bar{\pi}_{H,0}$ and $\bar{\pi}^*_{F,0}$ have to be interpreted as function of the predetermined or exogenous variables that will be self-consistent in the equilibrium considered. In particular we note that

$$u_t = [\xi_1 \hat{a}_{W,t} + \xi_2 \hat{\mu}_{W,t} + \xi_3 \hat{G}_{W,t}] + (1-n)[\gamma_1 \hat{a}_{R,t} + \gamma_2 \hat{\mu}_{R,t} + \gamma_3 \hat{G}_{R,t}], \tag{62}$$

$$u_t^* = [\xi_1 \hat{a}_{W,t} + \xi_2 \hat{\mu}_{W,t} + \xi_3 \hat{G}_{W,t}] - n[\gamma_1 \hat{a}_{R,t} + \gamma_2 \hat{\mu}_{R,t} + \gamma_3 \hat{G}_{R,t}], \tag{63}$$

where

$$\xi_1 \equiv \frac{\eta \bar{\mu}^{-1} (\bar{\mu} - 1) \rho (1 - s_c)}{\lambda_c^w (s_c \eta + \rho)^2},$$

$$\xi_2 \equiv \frac{\bar{\mu}^{-1} (s_c \eta + \rho)^2 - \bar{\mu}^{-1} (\bar{\mu} - 1) \rho (1 - s_c)}{\lambda_c^w (s_c \eta + \rho)^2},$$

$$\xi_3 \equiv -\frac{\bar{\mu}^{-1} (\bar{\mu} - 1) \rho (\rho + \eta)}{\lambda_c^w (s_c \eta + \rho)^2},$$

$$\gamma_1 \equiv \frac{n(1-n) \eta \theta \bar{\mu}^{-1} (\bar{\mu} - 1) (1 - s_c)}{\tilde{\lambda}_q^w (s_c \eta + \rho)^2}$$

$$\gamma_2 \equiv \frac{n(1-n)\theta [\bar{\mu}^{-1} (1 + \eta \theta s_c)(s_c \eta + \rho) - \bar{\mu}^{-1} (\bar{\mu} - 1)(1 - s_c)]}{\tilde{\lambda}_q^w (s_c \eta + \rho)^2},$$

$$\gamma_3 \equiv -\frac{n(1-n) \bar{\mu}^{-1} (\bar{\mu} - 1)(1 + \theta \eta)}{\tilde{\lambda}_q^w (s_c \eta + \rho)^2}.$$

Analysis of the gains from cooperation (section 3.2 and section 4 of the paper)

This subsection presents the details of the results of section 4 of the paper. In a non-cooperative equilibrium, each country minimizes its loss function by choosing its path of GDP inflation as a function of the shock, taking as given the strategy of the other policymaker. In particular, in a non-cooperative equilibrium where each country commits from a timeless perspective, the policymaker in country H minimizes the loss function (49) under the constraints (59)–(61) and the constraint that $\pi_{H,0} = \bar{\pi}_H$.

The lagrangian of this problem can be written as

$$\begin{aligned}
\mathcal{L} &= E_0\sum_{t=0}^{\infty}\beta^t[\frac{1}{2}\lambda_{y_h}(\hat{Y}_{H,t}-\tilde{Y}^h_{H,t})^2+\frac{1}{2}\lambda_{y_f}(\hat{Y}^*_{F,t}-\tilde{Y}^h_{F,t})^2+\frac{1}{2}\lambda_q(\tilde{T}_t-\tilde{T}^h_t)^2+\frac{1}{2}\lambda_{\pi_h}\pi^2_{H,t} \\
&\quad +\frac{1}{2}\lambda_{\pi_f}\pi^{*2}_{F,t}]+\varphi_{1,t}[\kappa^{-1}\pi_{H,t}-(\hat{Y}_{H,t}-\tilde{Y}^h_{H,t})-(1-n)\psi(\tilde{T}_t-\tilde{T}^h_t)-\beta\kappa^{-1}\pi_{H,t+1}]+ \\
&\quad +\varphi_{2,t}[\kappa^{*-1}\pi^*_{F,t}-(\hat{Y}^*_{F,t}-\tilde{Y}^h_{F,t})+n\psi(\tilde{T}_t-\tilde{T}^h_t)-\beta\kappa^{*-1}\pi^*_{F,t+1}]+n(1-n)\varphi_{3,t}[(\tilde{T}_t-\tilde{T}^h_t)+ \\
&\quad -\theta^{-1}s_c^{-1}(\hat{Y}_{H,t}-\tilde{Y}^h_{H,t})+\theta^{-1}s_c^{-1}(\hat{Y}^*_{F,t}-\tilde{Y}^h_{F,t})]-\varphi_{1,-1}\kappa^{-1}\pi_{H,0}.
\end{aligned}$$

The first-order conditions are

$$\lambda_{y_h}(\hat{Y}_{H,t}-\tilde{Y}^h_{H,t})-\varphi_{1,t}-\theta^{-1}s_c^{-1}n(1-n)\varphi_{3,t}=0, \tag{64}$$

$$\lambda_{y_f}(\hat{Y}^*_{F,t}-\tilde{Y}^h_{F,t})-\varphi_{2,t}+\theta^{-1}s_c^{-1}n(1-n)\varphi_{3,t}=0, \tag{65}$$

$$\lambda_q(\tilde{T}_t-\tilde{T}^h_t)-(1-n)\psi\varphi_{1,t}+n\psi\varphi_{2,t}+n(1-n)\varphi_{3,t}=0, \tag{66}$$

$$\lambda_{\pi_h}\pi_{H,t}+\kappa^{-1}(\varphi_{1,t}-\varphi_{1,t-1})=0. \tag{67}$$

We now define $\tilde{\varphi}_{1,t}\equiv\varphi_{1,t}/n$ and $\tilde{\varphi}_{2,t}\equiv\varphi_{2,t}/(1-n)$, $\tilde{\lambda}_{y_h}\equiv\lambda_{y_h}/n$, $\tilde{\lambda}_{y_f}\equiv\lambda_{y_f}/(1-n)$ and $\tilde{\lambda}_a\equiv\lambda_q/(n(1-n))$. We can then combine (64), (65) to get

$$n\tilde{\lambda}_{y_h}(\hat{Y}_{H,t}-\tilde{Y}^h_{H,t})+(1-n)\tilde{\lambda}_{y_f}(\hat{Y}^*_{F,t}-\tilde{Y}^h_{F,t})-n\tilde{\varphi}_{1,t}-(1-n)\tilde{\varphi}_{2,t}=0. \tag{68}$$

We take also the difference of (64) and (65) and substitute in (66) for $\varphi_{3,t}$ obtaining

$$\tilde{\lambda}_{y_h}(\hat{Y}_{H,t}-\tilde{Y}^h_{H,t})-\tilde{\lambda}_{y_f}(\hat{Y}^*_{F,t}-\tilde{Y}^h_{F,t})-(1+\theta^{-1}s_c^{-1}\psi)(\tilde{\varphi}_{1,t}-\tilde{\varphi}_{2,t})+\theta^{-1}s_c^{-1}\tilde{\lambda}_a(\tilde{T}_t-\tilde{T}^h_t)=0. \tag{69}$$

We can substitute (69) into (68) obtaining

$$\begin{aligned}
\tilde{\varphi}_{1,t} &= [n+(1-n)(1+\theta^{-1}s_c^{-1}\psi)^{-1}]\tilde{\lambda}_{y_h}(\hat{Y}_{H,t}-\tilde{Y}^h_{H,t})+ \\
&\quad +(1-n)\tilde{\lambda}_{y_f}\theta^{-1}s_c^{-1}\psi(1+\theta^{-1}s_c^{-1}\psi)^{-1}(\hat{Y}^*_{F,t}-\tilde{Y}^h_{F,t}) \\
&\quad +(1-n)\theta^{-1}s_c^{-1}\tilde{\lambda}_a(1+\theta^{-1}s_c^{-1}\psi)^{-1}(\hat{T}_t-\tilde{T}^h_t)
\end{aligned} \tag{70}$$

We note that

$$\begin{aligned}
(\hat{Y}^*_{F,t}-\tilde{Y}^h_{F,t}) &= \hat{Y}_{H,t}-s_c\theta\hat{T}_t-(\hat{G}_{H,t}-\hat{G}_{F,t})-\tilde{Y}^h_{F,t} \\
&= (\hat{Y}_{H,t}-\tilde{Y}^h_{H,t})-s_c\theta(\hat{T}_t-\tilde{T}^h_t)+(\tilde{Y}^h_{H,t}-\tilde{Y}^h_{F,t}-s_c\theta\tilde{T}^h_t-(\hat{G}_{H,t}-\hat{G}_{F,t}))
\end{aligned}$$

We can then substitute into (70) obtaining

$$\tilde{\varphi}_{1,t}=\vartheta_1(\hat{Y}_{H,t}-\tilde{Y}^h_{H,t})+\vartheta_2(\hat{T}_t-\tilde{T}^h_t)+\vartheta_3\mathcal{A}_t, \tag{71}$$

where

$$\vartheta_1 \equiv [n+(1-n)(1+\theta^{-1}s_c^{-1}\psi)^{-1}]\tilde{\lambda}_{y_h} + (1-n)\tilde{\lambda}_{y_f}\theta^{-1}s_c^{-1}\psi(1+\theta^{-1}s_c^{-1}\psi)^{-1},$$

$$\vartheta_2 \equiv (1-n)\theta^{-1}s_c^{-1}\tilde{\lambda}_a(1+\theta^{-1}s_c^{-1}\psi)^{-1} - \theta(1-n)\tilde{\lambda}_{y_f}\theta^{-1}\psi(1+\theta^{-1}s_c^{-1}\psi)^{-1},$$

$$\vartheta_3 \equiv (1-n)\tilde{\lambda}_{y_f}\theta^{-1}s_c^{-1}\psi(1+\theta^{-1}s_c^{-1}\psi)^{-1},$$

$$\mathcal{A}_t \equiv (\tilde{Y}_{H,t}^h - \tilde{Y}_{F,t}^h - s_c\theta\tilde{T}_t^h - (\hat{G}_{H,t} - \hat{G}_{F,t})).$$

Finally, we can combine (71) with (67) obtaining

$$\kappa\tilde{\lambda}_{\pi_h}\pi_{H,t} + \vartheta_1\Delta(\hat{Y}_{H,t} - \tilde{Y}_{H,t}^h) + \vartheta_2\Delta(\hat{T}_t - \tilde{T}_t^h) + \vartheta_3\Delta\mathcal{A}_t = 0. \tag{72}$$

We can repeat the same steps for the foreign country obtaining a rule of the form

$$\kappa^*\tilde{\lambda}_{\pi_f}^*\pi_{F,t}^* + \vartheta_1^*\Delta(\hat{Y}_{F,t}^* - \tilde{Y}_{F,t}^f) + \vartheta_2^*\Delta(\hat{T}_t - \tilde{T}_t^f) + \vartheta_3^*\Delta\mathcal{A}_t^* = 0, \tag{73}$$

for certain parameters $\tilde{\lambda}_{\pi_f}^*$, ϑ_1^*, ϑ_2^*, ϑ_3^* and variable $\mathcal{A}_t^*$. We finally recall from the analysis of section 4 in the text, that the two targeting rules that implement the cooperative solution have the following form

$$\kappa\lambda_{\pi_h}^w\pi_{H,t} + \lambda_y^w\Delta(\hat{Y}_{H,t} - \tilde{Y}_{H,t}^w) - (1-n)\gamma\Delta(\hat{T}_t - \tilde{T}_t^w) = 0, \tag{74}$$

$$\kappa^*\lambda_{\pi_f}^w\pi_{F,t}^* + \lambda_y^w\Delta(\hat{Y}_{F,t}^* - \tilde{Y}_{F,t}^w) + n\gamma\Delta(\hat{T}_t - \tilde{T}_t^w) = 0. \tag{75}$$

where $\gamma \equiv \psi\bar{\mu}^{-1}s_c^{-1}\eta(\bar{\mu}-1)(1-s_c)(s_c\eta+\theta^{-1})^{-1}$. To study under which conditions there are no gains from cooperation, we study when the targeting rules (72) and (73) determine the same equilibrium as the targeting rules (74) and (75). In the first two cases, indeed, the targeting rules (72) and (73) coincide with the targeting rules (74) and (75), respectively.

Case I: $\theta = 1$, $s_c = 1$, with only productivity shocks $\hat{a}_t$ and $\hat{a}_t^*$.

A first case in which the targeting rules coincide and then there are no gains from cooperation is when the loss functions, L and L^*, coincide. This is the case when $L_t = L_t^* = L_t^w$, i.e when $L_t^R = 0$ for each t. In particular $L_t^R = 0$ if and only if $s_c = 1$, $\theta = 1$ and there are only productivity shocks $\hat{a}_t$ and $\hat{a}_t^*$. It is then the case that under these conditions (72) and (73) coincide with (74) and (75).

Case II: $\theta\rho = 1$, $s_c = 1$, with only productivity shocks, $\hat{a}_t$ and $\hat{a}_t^*$.

The second case in which there are no gains from cooperation corresponds to the case of independent economies. Under productivity shocks and symmetric demand shocks, there are no gains from cooperation when $s_c = 1$ and $\rho\theta = 1$.

First we show that when $\rho\theta = 1$ it is possible to write the loss function for the cooperative problem as

$$L_t^W = n\lambda_y^w(\hat{Y}_{H,t} - \tilde{Y}_{H,t}^w)^2 + (1-n)\lambda_y^w(\hat{Y}_{F,t}^* - \tilde{Y}_{F,t}^w)^2 + \lambda_{\pi_h}^w \pi_{H,t}^2 + \lambda_{\pi_f}^w \pi_{F,t}^{*2}.$$

Moreover we note that L_t^R can be written as

$$L_t^R = \lambda_{y_h}^R(\hat{Y}_{H,t} - \tilde{Y}_{H,t}^R)^2 - \lambda_{y_f}^R(\hat{Y}_{F,t}^* - \tilde{Y}_{F,t}^R)^2 + \lambda_{\pi_h}^R \pi_{H,t}^2 + \lambda_{\pi_f}^R \pi_{F,t}^2$$

We can then write L_t and L_t^* as

$$L_t = \lambda_{y_h}(\hat{Y}_{H,t} - \tilde{Y}_{H,t}^h)^2 + \lambda_{y_f}(\hat{Y}_{F,t}^* - \tilde{Y}_{F,t}^h)^2 + \lambda_{\pi_h}\pi_{H,t}^2 + \lambda_{\pi_f}\pi_{F,t}^{*2},$$

$$L_t^* = \lambda_{y_h}^*(\hat{Y}_{H,t} - \tilde{Y}_{H,t}^f)^2 + \lambda_{y_f}^*(\hat{Y}_{F,t}^* - \tilde{Y}_{F,t}^f)^2 + \lambda_{\pi_h}^*\pi_{H,t}^2 + \lambda_{\pi_f}^*\pi_{F,t}^{*2},$$

where now

$$\tilde{\tilde{Y}}{}_{H,t}^h \equiv (\lambda_{y_h})^{-1}(n\lambda_{y_h}^w \tilde{Y}_{H,t}^w + (1-n)\lambda_{y_h}^R \tilde{Y}_{H,t}^R),$$
$$\tilde{\tilde{Y}}{}_{F,t}^h \equiv (\lambda_{y_f})^{-1}(1-n)(\lambda_{y_f}^w \tilde{Y}_{F,t}^w - \lambda_{y_f}^R \tilde{Y}_{F,t}^R),$$

$$\tilde{\tilde{Y}}{}_{H,t}^f \equiv (\lambda_{y_h}^*)^{-1}(n\lambda_{y_h}^w \tilde{Y}_{H,t}^w - n\lambda_{y_h}^R \tilde{\tilde{Y}}{}_{H,t}^R),$$
$$\tilde{\tilde{Y}}{}_{F,t}^f \equiv (\lambda_{y_f}^*)^{-1}[(1-n)\lambda_{y_f}^w \tilde{Y}_{F,t}^w + n\lambda_{y_f}^R \tilde{\tilde{Y}}{}_{F,t}^R].$$

It is now there case that the targeting rules in the non-cooperative equilibrium have the form

$$\kappa\lambda_{\pi_h}\pi_{H,t} + \lambda_{y_h}\Delta(\hat{Y}_{H,t} - \tilde{\tilde{Y}}{}_{H,t}^h) = 0,$$
$$\kappa^*\lambda_{\pi_f}^*\pi_{F,t}^* + \lambda_{y_f}^*\Delta(\hat{Y}_{F,t}^* - \tilde{\tilde{Y}}{}_{F,t}^h) = 0,$$

while in the cooperative solution they have the form

$$\kappa\lambda_{\pi_h}^w\pi_{H,t} + \lambda_y^w\Delta(\hat{Y}_{H,t} - \tilde{Y}_{H,t}^w) = 0,$$

$$\kappa^*\lambda_{\pi_f}^w\pi_{F,t}^* + \lambda_y^w\Delta(\hat{Y}_{F,t}^* - \tilde{Y}_{F,t}^w) = 0.$$

It is clear that under the assumption $\theta\rho = 1$ there are no gains from coordination if and only if $\tilde{\tilde{Y}}{}_{H,t}^h = \tilde{Y}_{H,t}^w$, $\tilde{\tilde{Y}}{}_{F,t}^h = \tilde{Y}_{F,t}^w$, $\lambda_{y_h} = \lambda_y^w$, $\lambda_{y_f} = \lambda_y^w$, $\lambda_{\pi_h} = \lambda_{\pi_h}^w$, $\lambda_{\pi_f}^* = \lambda_{\pi_f}^w$. For this to be the case it should be that $s_c = 1$, with only productivity shocks $\hat{a}_t$ and $\hat{a}_t^*$.

Case III: $s_c = 1$, with only symmetric productivity and demand shocks $\hat{a}_t = \hat{a}_t^*$, $\hat{g}_t = \hat{g}_t^*$.

Under these assumptions the central planner loss function boils down to

$$L_t^W = n\lambda_y^w(\hat{Y}_{H,t} - \tilde{Y}_t^w)^2 + (1-n)\lambda_y^w(\hat{Y}_{F,t}^* - \tilde{Y}_t^w)^2 + n(1-n)\lambda_q^w\hat{T}_t^2 + n\lambda_{\pi_h}^w\pi_{H,t}^2 + (1-n)\lambda_{\pi_f}^w\pi_{F,t}^{*2},$$

where

$$\tilde{Y}_t^w \equiv \frac{\eta}{\eta+\rho}\hat{a}_{W,t},$$

while the structural equilibrium conditions collapse to

$$\pi_{H,t} = \kappa[(\hat{Y}_{H,t} - \tilde{Y}_t^w) + (1-n)\psi\hat{T}_t] + \beta E_t\pi_{H,t+1},$$

$$\pi_{F,t}^* = \kappa^*[(\hat{Y}_{F,t}^* - \tilde{Y}_t^w) - n\psi\hat{T}_t] + \beta E_t\pi_{F,t+1}^*,$$

$$\hat{T}_t = \theta^{-1}s_c^{-1}(\hat{Y}_{H,t} - \hat{Y}_{F,t}^*).$$

It is then the case that the targets $\pi_{H,t} = 0$ and $\pi_{F,t}^* = 0$ implement the optimal cooperative equilibrium. It can be also shown that the loss functions L_t and L_t^* can be written as

$$L_t = \lambda_{y_h}(\hat{Y}_{H,t} - \tilde{Y}_t^w)^2 + \lambda_{y_f}(\hat{Y}_{F,t}^* - \tilde{Y}_t^w)^2 + \lambda_q\hat{T}_t^2 + \lambda_{\pi_h}\pi_{H,t}^2 + \lambda_{\pi_f}\pi_{F,t}^{*2}$$

$$L_t^* = \lambda_{y_h}^*(\hat{Y}_{H,t} - \tilde{Y}_t^w)^2 + \lambda_{y_f}^*(\hat{Y}_{F,t}^* - \tilde{Y}_t^w)^2 + \lambda_q^*\hat{T}_t^2 + \lambda_{\pi_h}^*\pi_{H,t}^2 + \lambda_{\pi_f}^*\pi_{F,t}^{*2},$$

from which it follows that the targets $\pi_{H,t} = 0$ and $\pi_{F,t}^* = 0$ are also a Nash equilibrium. Thus there are no gains from cooperation in this case.

Analysis of the optimal cooperative allocation in the general case (section 4 of the main text)

In a cooperative equilibrium, each country minimizes the social loss function by choosing its path of GDP inflation as a function of the shocks. In particular, in a cooperative equilibrium where each country commits from a timeless perspective, the policymakers in country H and F minimize the loss function (41) under the constraints (59)–(61) and the constraint that $\pi_{H,0} = \bar{\pi}_{H,0}$ and $\pi_{F,0}^* = \bar{\pi}_{F,0}^*$

$$\begin{aligned}\mathcal{L} = E_0\sum_{t=0}^{\infty}\beta^t[&\frac{1}{2}n\lambda_y^w y_{H,t}^2 + \frac{1}{2}(1-n)\lambda_y^w y_{F,t}^{*2} + \frac{1}{2}n(1-n)\lambda_q^w q_t^2 \\ &+ \frac{1}{2}n\lambda_{\pi_h}^w\pi_{H,t}^2 + \frac{1}{2}(1-n)\lambda_{\pi_f}^w\pi_{F,t}^{*2}] + n\varphi_{1,t}[\kappa^{-1}\pi_{H,t} - y_{H,t} - (1-n)\psi q_t - \beta\kappa^{-1}\pi_{H,t+1}] + \\ &+ (1-n)\varphi_{2,t}[\kappa^{*-1}\pi_{F,t}^* - y_{F,t}^* + n\psi q_t - \beta\kappa^{*-1}\pi_{F,t+1}^*] + n(1-n)\varphi_{3,t}[q_t + \\ &- \theta^{-1}s_c^{-1}y_{H,t} + \theta^{-1}s_c^{-1}y_{F,t}^*] - n\varphi_{1,-1}\kappa^{-1}\pi_{H,0} - (1-n)\varphi_{2,-1}\kappa^{*-1}\pi_{F,0}^*\end{aligned}$$

with $y_{H,t} \equiv (\hat{Y}_{H,t} - \tilde{Y}^w_{H,t})$, $y^*_{F,t} \equiv (\hat{Y}^*_{F,t} - \tilde{Y}^w_{F,t})$ and $q_t \equiv (\hat{T}_t - \tilde{T}^w_t)$ and where $\varphi_{1,t}, \varphi_{2,t}$ and $\varphi_{3,t}$ are the lagrangian multipliers associated with (59), (60) and (61) respectively. $\varphi_{1,-1}$, $\varphi_{2,-1}$ are the multipliers associated with the initial conditions $\pi_{H,0}$ and $\pi^*_{F,0}$.

The first-order condition with respect to $y_{H,t}$, $y^*_{F,t}$ and q_t are

$$\lambda^w_y y_{H,t} = \varphi_{1,t} + (1-n)\theta^{-1} s_c^{-1} \varphi_{3,t}, \tag{76}$$

$$\lambda^w_y y^*_{F,t} = \varphi_{2,t} - n\theta^{-1} s_c^{-1} \varphi_{3,t}, \tag{77}$$

$$\lambda^w_q q_t = \psi \varphi_{1,t} - \psi \varphi_{2,t} - \varphi_{3,t}, \tag{78}$$

while the ones with respect to $\pi_{H,t}$ and $\pi^*_{F,t}$ are

$$\kappa \lambda^w_{\pi_h} \pi_{H,t} = \varphi_{1,t} - \varphi_{1,t-1}, \tag{79}$$

$$\kappa^* \lambda^w_{\pi_f} \pi^*_{F,t} = \varphi_{2,t} - \varphi_{2,t-1}, \tag{80}$$

for each $t \geq 0$.

We first characterize the properties of the optimal cooperative outcome. By taking a weighted average with weights n and $(1-n)$ of (76) and (77), we obtain

$$\lambda^w_y [n y_{H,t} + (1-n) y^*_{F,t}] = n\varphi_{1,t} + (1-n)\varphi_{2,t}. \tag{81}$$

We then take the difference of (76) and (77) and combine it with (78), obtaining

$$(\lambda^w_y + \theta^{-2} s_c^{-2} \lambda^w_q)(y_{H,t} - y^*_{F,t}) = (1 + \theta^{-1} s_c^{-1} \psi)(\varphi_{1,t} - \varphi_{2,t}), \tag{82}$$

where we have used the fact that $y_{H,t} - y^*_{F,t} = \theta s_c q_t$. We further note that we can write (82) as

$$\lambda^w_y (y_{H,t} - y^*_{F,t}) - \gamma q_t = (\varphi_{1,t} - \varphi_{2,t}) \tag{83}$$

where we have used the relation $\lambda^w_q = \theta s_c^{-1} \psi [s_c^2 \lambda^w_y - \bar{\mu}^{-1}(\bar{\mu}-1)(1-s_c) s_c \eta (s_c \eta + \rho)^{-1}]$ and defined γ as $\gamma \equiv \psi \bar{\mu}^{-1} s_c^{-1} \eta (\bar{\mu}-1)(1-s_c)(s_c \eta + \theta^{-1})^{-1}$.

By using (81) and (83), we can obtain

$$\lambda^w_y y_{H,t} - (1-n)\gamma q_t = \varphi_{1,t},$$

$$\lambda^w_y y^*_{F,t} + n\gamma q_t = \varphi_{2,t},$$

which combined with (79) and (80) yields the following relation

$$\kappa \lambda^w_{\pi_h} \pi_{H,t} + \lambda^w_y \Delta y_{H,t} - (1-n)\gamma \Delta q_t = 0, \tag{84}$$

$$\kappa^* \lambda^w_{\pi_f} \pi^*_{F,t} + \lambda^w_y \Delta y^*_{F,t} + n\gamma \Delta q_t = 0. \tag{85}$$

Optimality of the flexible price allocation (section 3 and 4 of the main text)

By inspecting (41), (59), (60) and (61), we have that when $u_t = u_t^* = 0$ it is optimal to set $\pi_{H,t} = \pi_{F,t}^* = 0 \; \forall t > 0$. This occurs under the following combinations of shocks and parameters:

Symmetric Shocks: a) $s_c = 1$ with symmetric productivity shocks $\hat{a}_t = \hat{a}_t^*$;
b) $\overline{\mu} = 1$ with symmetric productivity and public expenditure shocks $\hat{a}_t = \hat{a}_t^*$, $\hat{G}_t = \hat{G}_t^*$.

In this case it is easy to check that from (62) and (63) $\xi_1 = 0$, $\xi_2 = 0$ and $\xi_4 = 0$. In general symmetric mark-up shocks imply a departure from the flexible price allocation.

Asymmetric Shocks: a) $s_c = 1$ with asymmetric productivity shocks $\hat{a}_t$ and $\hat{a}_t^*$.
b) $\overline{\mu} = 1$ with asymmetric productivity and public expenditure shocks, i.e. $\hat{a}_t, \hat{a}_t^*$, and $\hat{G}_t, \hat{G}_t^*$.

Again by inspection of (62) and (63), we obtain $\gamma_1 = 0$ and $\gamma_3 = 0$ depending on the cases.

Properties of the Nominal Exchange Rate under Cooperation (section 3 of the main text)

First we note that $\kappa\lambda_{\pi_h}^w = \kappa^*\lambda_{\pi_f}^w$. We now use (82) with $y_{H,t} - y_{F,t}^* = \theta s_c q_t$ to get that

$$q_t = \frac{(1 + \theta^{-1} s_c^{-1} \psi)}{(\lambda_y^w + \theta^{-2} s_c^{-2} \lambda_q^w)\theta s_c}(\varphi_{1,t} - \varphi_{2,t}).$$

By combining the previous equation with (79) and (80) and with the law of motion of the terms of trade we get that

$$s_t = \left(\frac{1}{\kappa\lambda_{\pi_h}^w} - \frac{(1 + \theta^{-1} s_c^{-1} \psi)}{(\lambda_y^w + \theta^{-2} s_c^{-2} \lambda_q^w)\theta s_c} \right)(\varphi_{1,t} - \varphi_{2,t}) + \tilde{T}_t^w.$$

When $\overline{\mu} = 1$ we obtain that

$$s_t = \left(\frac{1}{\sigma} - \frac{1}{\theta s_c} \right) \frac{s_c^2}{(\rho + \eta s_c)}(\varphi_{1,t} - \varphi_{2,t}) + \frac{\eta}{1 + \theta s_c \eta}\left(\hat{a}_{R,t} - \hat{G}_{R,t} \right).$$

Note that in this case $u_t = \hat{\mu}_t$ and $u_t^* = \hat{\mu}_t^*$. It is easy to see that when there are only mark-up shocks then a fixed exchange rate regime would be optimal when $\frac{1}{\sigma} = \frac{1}{\theta s_c}$. When there are no mark-up shocks it then follows htat

$$s_t = \frac{\eta}{1 + \theta s_c \eta}\left(\hat{a}_{R,t} - \hat{G}_{R,t} \right).$$

On the other hand when $s_c = 1$ and $\overline{\mu} > 1$

$$s_t = \left(\frac{1}{\sigma} - \frac{1}{\theta}\right) \frac{1}{(\eta+\rho) + \bar{\mu}^{-1}(\bar{\mu}-1)(1-\rho)} (\varphi_{1,t} - \varphi_{2,t}) + \tilde{T}_t^w$$

where $\tilde{T}_t^w$ is a combination of asymmetric productivity, mark-up and public expenditure shocks. Note that in this case $u_t = \xi_2 \hat{\mu}_t + \xi_3 \hat{G}_t$ and $u_t^* = \xi_2 \hat{\mu}_t^* + \xi_3 \hat{G}_t^*$. When there are no mark-up or government expenditure shocks then the nominal exchange rate follows

$$s_t = \frac{\eta}{1+\theta\eta} \hat{a}_{R,t}.$$

Determinacy of Optimal Cooperative Solution in the General Case ($\bar{\mu} \neq 1$) **(Section 4 of the main text)**

We show that the first-order conditions (76)–(80) combined with the constraints (59)–(61) and the initial conditions $\varphi_{1,-1}$ and $\varphi_{2,-1}$ yield to a determinate equilibrium. First we use (76)–(80) and (61) to write (59) and (60) in terms of only the lagrangian multipliers and the shocks as it follows

$$E_t \varphi_{1,t+1} = \left(1 + \frac{1}{\beta} + \frac{\vartheta_1 \xi \kappa}{\beta}\right) \varphi_{1,t} + \frac{(1-n)\vartheta_2 \xi \kappa}{\beta} \varphi_{2,t} - \frac{1}{\beta} \varphi_{1,t-1} + \frac{\xi \kappa}{\beta} u_t \tag{86}$$

$$E_t \varphi_{2,t+1} = \left(1 + \frac{1}{\beta} + \frac{\vartheta_3 \xi \kappa^*}{\beta}\right) \varphi_{2,t} + \frac{n\vartheta_2 \xi \kappa^*}{\beta} \varphi_{1,t} - \frac{1}{\beta} \varphi_{2,t-1} + \frac{\xi \kappa^*}{\beta} u_t^* \tag{87}$$

where

$$\vartheta_1 \equiv n\lambda_y^{-1} + (1-n)\tilde{\lambda}_q^{-1}(\theta s_c + \psi)^2,$$

$$\vartheta_2 \equiv \lambda_y^{-1} - \tilde{\lambda}_q^{-1}(\theta s_c + \psi)^2,$$

$$\vartheta_3 \equiv (1-n)\lambda_y^{-1} + n\tilde{\lambda}_q^{-1}(\theta s_c + \psi)^2,$$

$$\xi \equiv \kappa \lambda_{\pi_h}^w = \kappa^* \lambda_{\pi_f}^w,$$

$$\tilde{\lambda}_q \equiv \theta^2 s_c^2 \lambda_y + \lambda_q.$$

where $\lambda_{\pi_h}^w$, $\lambda_{\pi_f}^w$, λ_y, λ_q, $\tilde{\lambda}_q$ are defined in the technical appendix. In particular, under reasonable parameters' restriction, $\lambda_{\pi_h}^w > 0$, $\lambda_{\pi_f}^w > 0$, $\lambda_y > 0$, $\tilde{\lambda}_q > 0$ which imply that $\xi > 0$, $\vartheta_1 > 0$ and $\vartheta_3 > 0$. We can write (7) and (8) in the following form

$$E_t z_{t+1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & 0 \end{bmatrix} z_t + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \epsilon_t \tag{88}$$

where $z_t' \equiv [\varphi_t \ \varphi_{t-1}]$ and $\varphi_t \equiv [\varphi_{1,t} \ \varphi_{2,t}]$; $\epsilon_t' \equiv [u_t \ u_t^*]$, A_j with $j = 1, 2, 3$, and B_1 are two by two matrices. In particular

$$A \equiv \begin{bmatrix} A_1 & A_2 \\ A_3 & 0 \end{bmatrix}$$

$$A_1 \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A_2 \equiv \begin{bmatrix} -\beta^{-1} & 0 \\ 0 & -\beta^{-1} \end{bmatrix} \qquad A_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with

$$\begin{aligned} a_{11} &\equiv \left(1 + \frac{1}{\beta} + \frac{\vartheta_1 \xi \kappa}{\beta}\right) > 0 \\ a_{12} &\equiv \frac{(1-n)\vartheta_2 \xi \kappa}{\beta} \\ a_{21} &\equiv \frac{n \vartheta_2 \xi \kappa^*}{\beta} \\ a_{22} &\equiv \left(1 + \frac{1}{\beta} + \frac{\vartheta_3 \xi \kappa^*}{\beta}\right) > 0 \end{aligned}$$

and B_1 is a block-diagonal matrix with elements $\xi\kappa$, $\xi\kappa^*$. In order to study determinacy, we need to inspect the roots of the characteristic polynomial associated with the matrix A which is

$$P(\psi) = \psi^4 - (a_{11} + a_{22})\psi^3 + (a_{11}a_{22} - a_{21}a_{12} + 2\beta^{-1})\psi^2 - (a_{11} + a_{22})\beta^{-1}\psi + \beta^{-2}.$$

First we note that

$$\psi_1 \psi_2 \psi_3 \psi_4 = \beta^{-2}, \tag{89}$$

$$\psi_1 + \psi_2 + \psi_3 + \psi_4 = a_{11} + a_{22} > 2(1 + \beta^{-1}); \tag{90}$$

moreover if $P(\psi) = 0$ then $P(\psi^{-1}\beta^{-1}) = 0$ so that we can further conclude that

$$\psi_1 \psi_2 = \beta^{-1} \qquad\qquad \psi_3 \psi_4 = \beta^{-1}. \tag{91}$$

Moreover, by Descartes sign rule all the roots are positive. We note that

$$\begin{aligned} P(1) &= (1 + \beta^{-1})^2 - (1 + \beta^{-1})(a_{11} + a_{22}) + a_{11}a_{22} - a_{21}a_{12} \\ &= \xi \lambda_y^{-1} \tilde{\lambda}_q^{-1} > 0 \end{aligned}$$

$$P(0) = \beta^{-2} > 0$$

The fact that all the roots are positive and that $P(1) > 0, P(0) > 0$ imply that there are either 0 or 2 real or complex roots or 4 complex roots within the unit circle. Conditions (90) and (91) exclude the first and latter possibilities. From conditions

(91), we can further conclude that the two roots are within the unit circle. The unique and stable solution of the system is obtained with the following steps. Let V the two by four matrix of left eigenvectors associated with the unstable roots. By pre-multiplying the system (88) with V we obtain

$$E_t k_{t+1} = \Lambda k_t + VB\epsilon_t \tag{92}$$

where Λ is a two by two diagonal matrix of the unstable eigenvalues on the diagonal and $k_t \equiv Vz_t$. The unique and stable solution to (92) is given by

$$k_t = -\sum_{j=0}^{\infty} \Lambda^{-j} VBE_t\epsilon_{t+j}$$

which implies that

$$\varphi_t = -V_1^{-1}V_2\varphi_{t-1} - V_1^{-1}\sum_{j=0}^{\infty} \Lambda^{-j} VBE_t\epsilon_{t+j} \tag{93}$$

where V_1 and V_2 are such that $V = [V_1 \ V_2]$. Equation (93) characterizes the optimal path of the vector φ_t given initial condition φ_{-1}; the paths for y_H, y_F^*, π_H, π_F^*, q_t can be derived using the conditions (76)–(80).

Targeting Rules in the General Case (Section 4)

To derive the desirable targeting rules for the general case we use the first-order conditions (76) to (80). First, we take a weighted average with weights n and $(1-n)$ of (76) and (77), obtaining

$$\lambda_y^w[ny_{H,t} + (1-n)y_{F,t}^*] = n\varphi_{1,t} + (1-n)\varphi_{2,t}. \tag{94}$$

We take the difference of (76) and (77) and combine it with (78), obtaining

$$(\lambda_y^w + \theta^{-2}s_c^{-2}\lambda_q^w)(y_{H,t} - y_{F,t}^*) = (1 + \theta^{-1}s_c^{-1}\psi)(\varphi_{1,t} - \varphi_{2,t}), \tag{95}$$

where we have used the fact that $y_{H,t} - y_{F,t}^* = \theta s_c q_t$. We further note that we can write (82) as

$$\lambda_y^w(y_{H,t} - y_{F,t}^*) - \gamma q_t = (\varphi_{1,t} - \varphi_{2,t}) \tag{96}$$

where we have used the relation $\lambda_q^w = \theta s_c^{-1}\psi[s_c^2\lambda_y^w - \bar{\mu}^{-1}(\bar{\mu}-1)(1-s_c)s_c\eta(s_c\eta + \rho)^{-1}]$ and defined γ as $\gamma \equiv \psi\bar{\mu}^{-1}s_c^{-1}\eta(\bar{\mu}-1)(1-s_c)(s_c\eta + \theta^{-1})^{-1}$. By using (81) and (83), we can obtain

$$\lambda_y^w y_{H,t} - (1-n)\gamma q_t = \varphi_{1,t},$$

$$\lambda_y^w y_{F,t}^* + n\gamma q_t = \varphi_{2,t},$$

which combined with (79) and (80) yields the following relation

$$\kappa\lambda_{\pi_h}^w \pi_{H,t} + \lambda_y^w \Delta y_{H,t} - (1-n)\gamma\Delta q_t = 0, \tag{97}$$

$$\kappa^*\lambda_{\pi_f}^w \pi_{F,t}^* + \lambda_y^w \Delta y_{F,t}^* + n\gamma\Delta q_t = 0. \tag{98}$$

We now use the following price relations, the terms of trade identity in first difference

$$\hat{T}_t = \hat{T}_{t-1} + \Delta S_t + \pi_{F,t}^* - \pi_{H,t}, \tag{99}$$

and the PPP as well in first difference

$$\pi_t = n\pi_{H,t} + (1-n)(\Delta S_t + \pi_{F,t}^*) = \Delta S_t + \pi_t^*. \tag{100}$$

Using (99) and (100), we can rewrite (74) and (75) as

$$(\kappa\lambda_{\pi_h}^w + \gamma)\pi_{H,t} + \lambda_y^w \Delta y_{H,t} - \gamma(\pi_t - \tilde{\pi}_t) = 0, \tag{101}$$

$$(\kappa^*\lambda_{\pi_f}^w + \gamma)\pi_{F,t}^* + \lambda_y^w \Delta y_{F,t}^* - \gamma(\pi_t^* - \tilde{\pi}_t^*) = 0, \tag{102}$$

where $\tilde{\pi}_t \equiv (1-n)(\tilde{T}_t^* - \tilde{T}_{t-1}^*)$ and $\tilde{\pi}_t^* \equiv -n(\tilde{T}_t^* - \tilde{T}_{t-1}^*)$.

Proof of determinacy of the solution implemented by the targeting rules in the general case

We now show that the targeting rules (101) and (102), combined with the conditions (99) and (100) and the constraints (59) to (61) yield to a determinate equilibrium that coincides with the optimal cooperative solution. We follow here an argument similar to Woodford (2003, ch. 6). It is easy to see that (101) and (102) combined with the conditions (99) and (100) imply (74) and (75). Let us define $\varphi_{1,t}$ and $\varphi_{2,t}$ for all $t \geq -1$ as

$$\varphi_{1,t} \equiv \lambda_y^w y_{H,t} - (1-n)\gamma q_t, \tag{103}$$

$$\varphi_{2,t} \equiv \lambda_y^w y_{F,t}^* + n\gamma q_t, \tag{104}$$

from which it follows that

$$\kappa\lambda_{\pi_h}^w \pi_{H,t} = -(\varphi_{1,t} - \varphi_{1,t-1}), \tag{105}$$

$$\kappa^*\lambda_{\pi_f}^w \pi_{F,t}^* = -(\varphi_{2,t} - \varphi_{2,t-1}). \tag{106}$$

Using (61) and (103)-(106), we can then retrieve the system of equations (76)–(80) which yields to a determinate equilibrium given the initial conditions

$$\varphi_{1,-1} \equiv \lambda_y^w y_{H,-1} - (1-n)\gamma q_{-1},$$

$$\varphi_{2,-1} \equiv \lambda_y^w y_{F,-1}^* + n\gamma q_{-1}.$$

Indeed the lagrangian multiplier $\varphi_{1,-1}$ and $\varphi_{2,-1}$ measure the commitment to expectations taken in periods before time 0. The timeless perspective optimal policy is the one that assigns a particular value to the commitment to expectations prior to period 0 such that the resulting optimal policy is time invariant.