Appendix for
Debt Deleveraging and The Exchange Rate

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1. Derivation of loss function (23)

To derive (23), we take a second-order approximation around the final steady state of the following Lagrangian

$$\begin{aligned}\mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \left\{ \xi \ln(C_t) + (1-\xi)\ln(C_t^*) + \lambda_{1,t}(Y_{H,t} - (\alpha T_t^{1-\alpha} C_t + (1-\alpha) T_t^{\alpha} C_t^*) + \right. \\ & \left. +\lambda_{2,t}(Y_{F,t}^* - ((1-\alpha)T_t^{-\alpha} C_t + \alpha T_t^{\alpha-1} C_t^*)) \right\}\end{aligned}$$

First it should be noted that in the steady state the following conditions

$$\xi \bar{C}^{-1} = \bar{T}^{1-\alpha}\bar{\lambda}_1, \tag{1}$$

$$(1-\xi)(\bar{C}^*)^{-1} = \bar{T}^{\alpha}\bar{\lambda}_1, \tag{2}$$

$$\bar{T}\bar{\lambda}_1 = \bar{\lambda}_2, \tag{3}$$

hold together with the two resource constraints.

By taking a second-order approximation of the above Lagrangian around the above-defined steady state, we obtain

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$$
\begin{aligned}
\mathcal{L} \;=\; & \sum_{t=0}^{\infty} \beta^t \left\{ \xi \bar{C}^{-1}(C_t - \bar{C}) - \xi \bar{C}^{-2}(C_t - \bar{C})^2 + (1-\xi)(\bar{C}^*)^{-1}(C_t^* - \bar{C}^*) + \right. \\
& -(1-\xi)(\bar{C}^*)^{-2}(C_t^* - \bar{C}^*)^2 - \bar{\lambda}_1[\alpha(1-\alpha)\bar{C}\bar{T}^{-\alpha}(T_t - \bar{T}) + \alpha\bar{T}^{1-\alpha}(C_t - \bar{C}) + \\
& +\alpha(1-\alpha)\bar{T}^{-\alpha}(T_t - \bar{T})(C_t - \bar{C}) - \alpha^2(1-\alpha)\bar{C}\bar{T}^{-\alpha-1}\frac{(T_t - \bar{T})^2}{2} + \\
& +\alpha(1-\alpha)\bar{C}^*\bar{T}^{\alpha-1}(T_t - \bar{T}) + (1-\alpha)\bar{T}^{\alpha}(C_t^* - \bar{C}^*) + \\
& \alpha(1-\alpha)\bar{T}^{\alpha-1}(T_t - \bar{T})(C_t^* - \bar{C}^*) - \alpha(1-\alpha)^2\bar{C}^*\bar{T}^{\alpha-2}\frac{(T_t - \bar{T})^2}{2}] + \\
& -\bar{\lambda}_2[-\alpha(1-\alpha)\bar{C}\bar{T}^{-\alpha-1}(T_t - \bar{T}) + (1-\alpha)\bar{T}^{-\alpha}(C_t - \bar{C}) + \\
& -\alpha(1-\alpha)\bar{T}^{-\alpha-1}(T_t - \bar{T})(C_t - \bar{C}) + \alpha(1+\alpha)(1-\alpha)\bar{C}\bar{T}^{-\alpha-2}\frac{(T_t - \bar{T})^2}{2} + \\
& +\alpha(\alpha-1)\bar{C}^*\bar{T}^{\alpha-2}(T_t - \bar{T}) + \alpha\bar{T}^{\alpha-1}(C_t^* - \bar{C}^*) + \alpha(\alpha-1)\bar{T}^{\alpha-2}(T_t - \bar{T})(C_t^* - \bar{C}^*) + \\
& \left. +\alpha(\alpha-1)(\alpha-2)\bar{C}^*\bar{T}^{\alpha-3}\frac{(T_t - \bar{T})^2}{2}] \right\}
\end{aligned}
$$

in which it should be noted that all the linear terms cancel out using the steady-state relationship. The second-order terms can be simplified and the above expression collapses to

$$
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ -\xi(\tilde{C}_t)^2 - (1-\xi)(\tilde{C}_t^*)^{-2} - \bar{\lambda}_1\alpha(1-\alpha)\bar{C}\bar{T}^{1-\alpha}\frac{\tilde{T}_t^2}{2} - \bar{\lambda}_1\alpha(1-\alpha)\bar{C}^*\bar{T}^{\alpha}\frac{\tilde{T}_t^2}{2} \right\}
$$

which can be further written as

$$
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ -\xi(\tilde{C}_t)^2 - (1-\xi)(\tilde{C}_t^*)^{-2} - \alpha(1-\alpha)\xi\frac{\tilde{T}_t^2}{2} - \alpha(1-\alpha)(1-\xi)\frac{\tilde{T}_t^2}{2} \right\}
$$

from which the loss function in the text follows.

2. Model equilibrium conditions

The model of Section 3 is represented by the following 18 equilibrium conditions

$$
(C_t^*)^{-\rho} = \beta E_t \left\{ (C_{t+1}^*)^{-\rho} \frac{(1+i_t)Q_t}{Q_{t+1}\Pi_{t+1}} \right\},
$$

$$
(C_t^*)^{-\rho} = \beta E_t \left\{ (C_{t+1}^*)^{-\rho} \frac{(1+i_t^*)}{\Pi_{t+1}^*} \right\},
$$

$$
(C_t)^{-\rho} \left\{ 1 - (1+i_t)\psi\left(\frac{d_t}{k_t}\right) \right\} = \beta E_t \left\{ (C_{t+1})^{-\rho} \frac{(1+i_t)}{\Pi_{t+1}} \right\},
$$

$$C_t = p_{H,t}Y_{H,t} + \frac{d_t}{(1+i_t)} - \frac{d_{t-1}}{\Pi_t} - k_t\chi\left(\frac{d_t}{k_t}\right)$$

$$Y^*_{F,t} = p_{F,t}^{-\theta}\left[(1-\alpha)C_t + \alpha Q_t^\theta C_t^*\right]$$

$$Y_{H,t} = p_{H,t}^{-\theta}\left[\alpha C_t + (1-\alpha)Q_t^\theta C_t^*\right]$$

$$p_{F,t}^{\theta-1} = \alpha\left(T_t\right)^{\theta-1} + (1-\alpha)$$

$$p_{H,t}^{\theta-1} = \alpha + (1-\alpha)\left(T_t\right)^{1-\theta}$$

$$\left(\frac{1-\lambda^*\left(\frac{\Pi^*_{F,t}}{\bar{\Pi}^*}\right)^{\tau-1}}{1-\lambda^*}\right)^{\frac{1+\eta\tau}{\tau-1}} = \frac{F_t^*}{K_t^*}$$

$$F_t^* = (C_t^*)^{-\rho}p_{F,t}\frac{1}{Q_t}Y^*_{F,t} + \beta\lambda^* E_t\left[F^*_{t+1}\left(\frac{\Pi^*_{F,t+1}}{\bar{\Pi}^*}\right)^{\tau-1}\right]$$

$$K_t^* = \tilde{\mu}(Y^*_{F,t})^{1+\eta} + \beta\lambda^* E_t\left[K^*_{t+1}\left(\frac{\Pi^*_{F,t+1}}{\bar{\Pi}^*}\right)^{\tau(1+\eta)}\right]$$

$$\left(\frac{1-\lambda\left(\frac{\Pi_{H,t}}{\bar{\Pi}}\right)^{\tau-1}}{1-\lambda}\right)^{\frac{1+\eta\tau}{\tau-1}} = \frac{F_t}{K_t}$$

$$F_t = (C_t)^{-\rho}p_{H,t}Y_{H,t} + \beta\lambda E_t\left[F_{t+1}\left(\frac{\Pi_{H,t+1}}{\bar{\Pi}}\right)^{\tau-1}\right]$$

$$K_t = \tilde{\mu}Y_{H,t}^{1+\eta} + \beta\lambda E_t\left[K_{t+1}\left(\frac{\Pi_{H,t+1}}{\bar{\Pi}}\right)^{\tau(1+\eta)}\right]$$

$$\frac{T_t}{T_{t-1}} = \frac{\Pi^*_{F,t}}{\Pi_{H,t}}\frac{S_t}{S_{t-1}}$$

$$Q_t = \left[(1-\alpha)p_{H,t}^{1-\theta} + \alpha p_{F,t}^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

$$\Pi_t = \frac{\left[\alpha\left(\Pi_{H,t}\right)^{1-\theta} + (1-\alpha)\left(T_t\Pi_{H,t}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}}{\left[\alpha + (1-\alpha)\left(T_{t-1}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}}$$

$$\Pi_t^* = \Pi_t\left(\frac{Q_t}{Q_{t-1}}\right)\left(\frac{S_{t-1}}{S_t}\right)$$

which need to be solved for the following 20 unknowns C_t, C_t^*, i_t, Q_t, Π_t, i_t^*, Π_t^*, T_t, $Y_{H,t}$,.$Y_{F,t}$, d_t, $\Pi^*_{F,t}$, F_t^*, K_t^*, $\Pi_{H,t}$, F_t, K_t, S_t/S_{t-1}, $p_{H,t}$, $p_{F,t}$ given the inflation targets $\bar{\Pi}^*$ and $\bar{\Pi}$ where two further restrictions come from the policy rules, specified in the text. Notice that $\tilde{\mu}$ is composite mark-up including the mark-ups in the goods and labor markets, i.e. $\tilde{\mu} = \mu \cdot \mu_w$ where $\mu_w \equiv \tau/(\tau - 1)$. We also have defined $p_{H,t} \equiv P_{H,t}/P_t$ and $p_{F,t} \equiv P_{F,t}/P_t$. Moreover, the zero-lower-bound constraint requires that $i_t \geq 0$ and $i_t^* \geq 0$. In the above equations, we have defined $\chi(d_t/k_t) = \tilde{\chi}(d_t/k, \bar{d}_t/k)$ since in equilibrium $\bar{d}_t = d_t$.

3. Model Solution

We define $y_t \equiv [z_t \; x_{t-1} \; w_t]$ as a vector of length n_y containing the control variables, z_t, of dimension n_z, the endogenous state variables, x_{t-1}, of dimension n_x and the exogenous state variables, w_t, of dimension n_w. In particular we may define vector of exogenous state variables more specifically, i.e.:

$$w_t \equiv \begin{bmatrix} k_t & i_t^z & i_t^{*z} & c_t & c_t^* & y_{H,t} & y_{F,t}^* & s_t \end{bmatrix}'$$

where k_t represents the safe level of debt, as defined in the main text; i_t^z and i_t^{*z} are two variables used to model the zero-lower bound on the nominal interest rates. Indeed in the log-linear approximation, the restriction that nominal interest rates should be above zero corresponds to have $\hat{\imath}_t \geq i_t^z$ and $\hat{\imath}_t^* \geq i_t^{*z}$. The variables c_t, c_t^*, $y_{H,t}$, $y_{F,t}^*$ and s_t are the defined in equation (14) and represent the log deviation between the final and initial steady state of C, C^*, Y_H, Y_F^* and T respectively.

Finally we define ϵ as a vector of length n_ϵ that collects innovations to the exogenous stochastic variables. Again we may define this vector more in detail:

$$\epsilon \equiv \begin{bmatrix} (\ln(k_{min}) - \ln(k_{max})) & -\ln(\frac{\Pi}{\beta} - 1) & -\ln(\frac{\Pi}{\beta} - 1) & \epsilon_c & \epsilon_{c^*} & \epsilon_{y_H} & \epsilon_{y_F^*} & \epsilon_s \end{bmatrix}'$$

where ϵ_x is defined as the log difference between the final and initial steady state for a generic variable X, i.e. $\epsilon_x \equiv \ln(\bar{X}) - \ln(X)$. The process for the exogenous state variables can be modeled as:

$$w_t = M_w w_{t-1} + \tilde{C}^t \; \epsilon$$

where M_w is an identity matrix of dimensions $n_w \times n_w$. $\tilde{C}^t$ is matrix of dimension $n_w \times n_\epsilon$ and it is an identity matrix when $t = 1$, otherwise it is a matrix of zeros.

We can write the model in a compact form as:

$$A \cdot y_{t+1} = B^t \cdot y_t + C^{t+1} \cdot \epsilon \tag{4}$$

where B^t and C^{t+1} are time-dependent matrices, A and B^t have dimension $n_y \times n_y$ and

C^{t+1} has dimension $n_y \times n_\epsilon$. The matrix C^t is of the form

$$C^{t+1} = \begin{bmatrix} H \\ \tilde{C}^{t+1} \end{bmatrix},$$

where H is a matrix of zeros of dimension $(n_y - n_w) \times n_\epsilon$.

We consider a framework which is flexible enough to treat the possibility that either Home interest rate, i_t, is at zero-lower bound, or Foreign interest rate, i_t^*, is at zero-lower bound, or both, or none of them. B^t should be adjusted accordingly depending on the different cases.

We define B^a as the matrix characterizing the case in which both interest rates are at zero lower bound; $B^H(B^F)$ is the matrix characterizing the case where only Home (Foreign) interest rate is at zero lower bound while B^n refers to the case where both interest rates are not constrained by the zero-lower bound.

In the model of Section (4), we verify the following sequence of events: from 0 to T_1 both interest rates are at zero lower bound (T_1 can also be 0), from T_1 to T_2 only the interest rate in country H is at zero lower bound. From T_2 onwards both interest rates are above zero. This timing implies that:

$$B^t = \begin{cases} B^a & for \;\; t \in (0; T_1] \\ B^H & for \;\; t \in (T_1, T_2] \\ B^n & for \;\; t \in (T_2, \infty] \end{cases}$$

where T_1 and T_2 are model specific and to be determined endogenously.

In the model of Section (6), we verify the following sequence of events: from 0 to $\tilde{T}_1$, the interest rate in country F is at the zero lower bound and, from $\tilde{T}_1$ onwards, both interest rates will be above zero.

This timing implies that:

$$B^t = \begin{cases} B^F & for \;\; t \in (0, \tilde{T}_1] \\ B^n & for \;\; t \in (\tilde{T}_1, \infty] \end{cases}$$

We can rewrite the system (4) by omitting the law of motion of the exogenous state variables:

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \end{bmatrix} \begin{bmatrix} z_{t+1} \\ x_t \end{bmatrix} = \begin{bmatrix} \tilde{B}_1^t & \tilde{B}_2^t & \tilde{B}_3^t \end{bmatrix} \begin{bmatrix} z_t \\ x_{t-1} \\ w_t \end{bmatrix} \tag{5}$$

where $\tilde{A}$ is a matrix of dimension $(n_y - n_w) \times (n_y - n_w)$ which is appropriately partitioned in the matrices $\tilde{A}_1$ and $\tilde{A}_2$, while $\tilde{B}$ is a matrix of dimension $(n_y - n_w) \times n_y$ which is appropriately partitioned in the matrices $\tilde{B}_1^t$, $\tilde{B}_2^t$, $\tilde{B}_3^t$.

We guess the following linear solution:

$$z_t = h_x^t x_{t-1} + h_w^t w_{t-1} + h_\epsilon^t \epsilon,$$

$$x_t = g_x^t x_{t-1} + g_w^t w_{t-1} + g_\epsilon^t \epsilon,$$

$$w_t = M_w w_{t-1} + \tilde{C}^t \epsilon,$$

We can plug the guessed solution into equation (5) and rearrange everything to get:

$$\begin{bmatrix} \tilde{A}_1 h_x^{t+1} + \tilde{A}_2 & -\tilde{B}_1^t \end{bmatrix} \begin{bmatrix} g_x^t \\ h_x^t \end{bmatrix} = \tilde{B}_2^t \tag{6}$$

$$\begin{bmatrix} \tilde{A}_1 h_x^{t+1} + \tilde{A}_2 & -\tilde{B}_1^t \end{bmatrix} \begin{bmatrix} g_w^t \\ h_w^t \end{bmatrix} = \tilde{B}_3^t M_w - \tilde{A}_1 h_w^{t+1} M_w \tag{7}$$

$$\begin{bmatrix} \tilde{A}_1 h_x^{t+1} + \tilde{A}_2 & -\tilde{B}_1^t \end{bmatrix} \begin{bmatrix} g_\epsilon^t \\ h_\epsilon^t \end{bmatrix} = \tilde{B}_3^t \tilde{C}^t - \tilde{A}_1 h_\epsilon^{t+1} - \tilde{A}_1 h_w^{t+1} \tilde{C}_t \tag{8}$$

Equations (6), (7) and (8) can be solved for the unknown matrices h_x^t, h_w^t, h_ϵ^t, g_x^t, g_w^t, g_ϵ^t working backward. Since we know that after T_2 (or $\tilde{T}_1$ in the model with foregn-denominated debt), there are no shocks and the interest rates are not constrained by the zero-lower bound, we can find the unknown time-invariant matrices h_x, h_w, h_ϵ, g_x, g_w, g_ϵ which applies for each $t \geq T_2$ (or $t \geq T_1$). Then starting from these matrices, we can get all the remaining matrices by using the above equations working backward. Given an initial guess on T_1, T_2 for one model and $\tilde{T}_1$ for the other model, we verify that the implied path of the nominal interest rates and the stay at the zero-lower bound are consistent with the guessed timing. Otherwise, we guess another T_1, T_2 or $\tilde{T}_1$, depending on the model.

4. Optimal policy

We take a second-order approximation of the welfare of world economy (25) around the final efficient steady state. First, notice that the objective can be written as

$$U_t = E_t \left\{ \sum_{t=0}^{\infty} \beta^t \left[\xi \left(\frac{C_t^{1-\rho}}{1-\rho} - \frac{Y_{H,t}^{1+\eta}}{1+\eta} \Delta_t \right) + (1-\xi) \left(\frac{C_t^{*1-\rho}}{1-\rho} - \frac{Y_{F,t}^{*1+\eta}}{1+\eta} \Delta_t^* \right) \right] \right\}$$

where the indexes of price dispersion are defined as

$$\Delta_t \equiv \lambda \left(\frac{\Pi_{H,t}}{\bar{\Pi}_t} \right)^{(1+\eta)\tau} \Delta_{t-1} + (1-\lambda) \left(\frac{1 - \lambda \left(\frac{\Pi_{H,t}}{\bar{\Pi}_t} \right)^{\tau-1}}{1-\lambda} \right)^{\frac{(1+\eta)\tau}{\tau-1}} \tag{9}$$

$$\Delta_t^* \equiv \lambda^* \left(\frac{\Pi_{F,t}}{\bar{\Pi}_t^*} \right)^{(1+\eta)\tau} \Delta_{t-1}^* + (1-\lambda^*) \left(\frac{1 - \lambda^* \left(\frac{\Pi_{F,t}}{\bar{\Pi}_t^*} \right)^{\tau-1}}{1-\lambda^*} \right)^{\frac{(1+\eta)\tau}{\tau-1}}. \tag{10}$$

A second-order approximation of the objective function around the efficient steady state delivers

$$\begin{aligned} U_t \;\; = \;\; & \bar{U} + E_t \left\{ \sum_{t=0}^{\infty} \beta^t [\xi[\bar{C}^{-\rho}(C_t - \bar{C}) - \bar{Y}_H^{\eta}(Y_{H,t} - \bar{Y}_H) - (1+\eta)^{-1}\bar{Y}_H^{1+\eta}(\Delta_t - 1) + \right. \\ & \frac{1}{2}\bar{C}^{-\rho-1}(C_t - \bar{C})^2 - \frac{1}{2}\bar{Y}_H^{\eta-1}(Y_{H,t} - \bar{Y}_H)^2] + (1-\xi)[\bar{C}^{*-\rho}(C_t^* - \bar{C}^*) + \\ & -\bar{Y}_F^{*\eta}(Y_{F,t}^* - \bar{Y}_F^*) - (1+\eta)^{-1}\bar{Y}_F^{*1+\eta}(\Delta_t^* - 1) + \frac{1}{2}\bar{C}^{*-\rho-1}(C_t^* - \bar{C}^*)^2 + \\ & -\frac{1}{2}\bar{Y}_F^{*\eta-1}(Y_{F,t}^* - \bar{Y}_F^*)^2]] + \mathcal{O}(\| \cdot \|^3) \end{aligned}$$

where $\mathcal{O}(\| \cdot \|^3)$ contains terms of order higher than the second. We take a second-order approximation of the constraints

$$Y_{F,t}^* = p_{F,t}^{-\theta} \left[(1-\alpha)C_t + \alpha Q_t^{\theta} C_t^* \right],$$

$$Y_{H,t} = p_{H,t}^{-\theta} \left[\alpha C_t + (1-\alpha) Q_t^{\theta} C_t^* \right],$$

considering that

$$\alpha p_{H,t}^{1-\theta} + (1-\alpha) p_{F,t}^{1-\theta} = 1,$$

$$Q_t^{1-\theta} = (1-\alpha) p_{H,t}^{1-\theta} + \alpha p_{F,t}^{1-\theta}.$$

where, consistently with Appendix B, we define $p_{H,t} \equiv P_{H,t}/P_t$ and $p_{F,t} \equiv P_{F,t}/P_t$.

Combining the second-order approximation of the constraints with the second-order approximation of the utility function at the efficient steady state, we can obtain after some steps that

$$
\begin{aligned}
U_t \;=\; & \bar{U} + \xi \bar{C}^{1-\rho} E_t \left\{ \sum_{t=0}^{\infty} \beta^t \left[-\rho \frac{\tilde{C}_t^2}{2} - \rho \frac{\bar{C}^* \bar{Q}}{\bar{C}} \frac{\tilde{C}_t^{*2}}{2} - \eta \frac{\bar{p}_H \bar{Y}_H}{\bar{C}} \frac{\tilde{Y}_{H,t}^2}{2} - \eta \frac{\bar{p}_F \bar{Y}_F^*}{\bar{C}} \frac{\tilde{Y}_{F,t}^{*2}}{2} \right.\right. \\
& -\theta \frac{\bar{p}_H \bar{Y}_H}{\bar{C}} \frac{\tilde{p}_{H,t}^2}{2} - \theta \frac{\bar{p}_F \bar{Y}_F^*}{\bar{C}} \frac{\tilde{p}_{F,t}^2}{2} + \theta \frac{\bar{C}^* \bar{Q}}{\bar{C}} \frac{\tilde{Q}_t^2}{2} \\
& \left.\left. -(1+\eta)^{-1} \frac{\bar{p}_H \bar{Y}_H}{\bar{C}} (\Delta_t - 1) - (1+\eta)^{-1} \frac{\bar{p}_F \bar{Y}_F^*}{\bar{C}} (\Delta_t^* - 1) \right] \right\} + \mathcal{O}(\|\cdot\|^3) \qquad (11)
\end{aligned}
$$

where we have transformed variables using the following relationship

$$
X_t = \bar{X} \left(1 + \tilde{X}_t + \frac{1}{2} \tilde{X}_t^2 \right) + \mathcal{O}(\|\cdot\|^3)
$$

for a generic variable X where $\tilde{X}$ denotes its log-deviation with respect to the final steady state. Notice that Δ_t and Δ_t^* in (11) are second-order terms which can be expressed in terms of the inflation rates by expanding through a second-order approximation (9) and (10). Using these approximations we can write (11) as

$$
\begin{aligned}
U_t \;=\; & \bar{U} + \xi \bar{C}^{1-\rho} E_t \left\{ \sum_{t=0}^{\infty} \beta^t \left[-\rho \frac{\tilde{C}_t^2}{2} - \rho \frac{\bar{C}^* \bar{Q}}{\bar{C}} \frac{\tilde{C}_t^{*2}}{2} - \eta \frac{\bar{p}_H \bar{Y}_H}{\bar{C}} \frac{\tilde{Y}_{H,t}^2}{2} - \eta \frac{\bar{p}_F \bar{Y}_F^*}{\bar{C}} \frac{\tilde{Y}_{F,t}^{*2}}{2} \right.\right. \\
& -\theta \frac{\bar{p}_H \bar{Y}_H}{\bar{C}} \frac{\tilde{p}_{H,t}^2}{2} - \theta \frac{\bar{p}_F \bar{Y}_F^*}{\bar{C}} \frac{\tilde{p}_{F,t}^2}{2} + \theta \frac{\bar{C}^* \bar{Q}}{\bar{C}} \frac{\tilde{Q}_t^2}{2} \\
& \left.\left. -\kappa \frac{\bar{p}_H \bar{Y}_H}{\bar{C}} \frac{(\pi_{H,t} - \bar{\pi})^2}{2} - \kappa^* \frac{\bar{p}_F \bar{Y}_F^*}{\bar{C}} \frac{(\pi_{F,t}^* - \bar{\pi}^*)^2}{2} \right] \right\} + \mathcal{O}(\|\cdot\|^3) \qquad (12)
\end{aligned}
$$

where

$$
\kappa \equiv \frac{\lambda \tau (1 + \eta \tau)}{(1-\lambda)(1-\lambda\beta)}, \qquad \kappa^* \equiv \frac{\lambda^* \tau (1 + \eta \tau)}{(1-\lambda^*)(1-\lambda^*\beta)},
$$

and $\pi_{H,t} \equiv \ln \Pi_{H,t}$, $\pi_{F,t}^* \equiv \ln \Pi_{F,t}^*$, $\bar{\pi} \equiv \ln \bar{\Pi}$ and $\bar{\pi}^* \equiv \ln \bar{\Pi}^*$.

The objective (12) can be written also in the equivalent form

$$
\begin{aligned}
U_t \;=\; & \bar{U} + \xi \bar{C}^{1-\rho} E_t \left\{ \sum_{t=0}^{\infty} \beta^t \left[-\rho \frac{(\hat{C}_t - c)^2}{2} - \rho \frac{\bar{C}^* \bar{Q}}{\bar{C}} \frac{(\hat{C}_t^* - c^*)^2}{2} - \eta \frac{\bar{p}_H \bar{Y}_H}{\bar{C}} \frac{(\hat{Y}_{H,t} - y_H)^2}{2} \right.\right. \\
& -\eta \frac{\bar{p}_F \bar{Y}_F^*}{\bar{C}} \frac{(\hat{Y}_{F,t}^* - y_F^*)^2}{2} - \theta \frac{\bar{p}_H \bar{Y}_H}{\bar{C}} \frac{(\hat{p}_{H,t} - p_H)^2}{2} - \theta \frac{\bar{p}_F \bar{Y}_F^*}{\bar{C}} \frac{(\hat{p}_{F,t} - p_F)^2}{2} + \theta \frac{\bar{C}^* \bar{Q}}{\bar{C}} \frac{(\hat{Q}_t - Q)^2}{2} \\
& \left.\left. -\kappa \frac{\bar{p}_H \bar{Y}_H}{\bar{C}} \frac{(\pi_{H,t} - \bar{\pi})^2}{2} - \kappa^* \frac{\bar{p}_F \bar{Y}_F^*}{\bar{C}} \frac{(\pi_{F,t}^* - \bar{\pi}^*)^2}{2} \right] \right\} + \mathcal{O}(\|\cdot\|^3) \qquad (13)
\end{aligned}
$$

where for a generic variable X, $\hat{X}$ denotes the log deviations with respect to the initial steady-state (before deleveraging) and x denotes the log difference between the final and initial steady state.

The objective function is now quadratic and can be appropriately evaluated by a log-linear approximation of the constraints around the initial steady state. By taking an approximation of the model equilibrium conditions presented in the above section of the Appendix, we respectively get

$$E_t\hat{C}^*_{t+1} = \hat{C}^*_t + \rho^{-1}[\hat{\imath}_t - E_t(\pi_{t+1} - \bar{\pi} + \hat{Q}_{t+1} - \hat{Q}_t)]$$

$$E_t\hat{C}^*_{t+1} = \hat{C}^*_t + \rho^{-1}[\hat{\imath}^*_t - E_t(\pi^*_{t+1} - \bar{\pi}^*)]$$

$$E_t\hat{C}_{t+1} = \hat{C}_t + \rho^{-1}[\hat{\imath}_t - E_t(\pi_{t+1} - \bar{\pi}) + \varpi_1(\hat{d}_t - \hat{k}_t)]$$

$$\hat{C}_t = \upsilon_1[\hat{p}_{H,t} + \hat{Y}_{H,t}] - \upsilon_2[\beta\hat{\imath}_t - (\pi_t - \bar{\pi})] + \upsilon_2\beta\hat{d}_t - \upsilon_2\hat{d}_{t-1} - \varpi_2(\hat{d}_t - \hat{k}_t)$$

$$\hat{Y}^*_{F,t} = -\theta\hat{p}_{F,t} + \upsilon_3\hat{C}_t + (1 - \upsilon_3)(\hat{C}^*_t + \theta\hat{Q}_t)$$

$$\hat{Y}_{H,t} = -\theta\hat{p}_{H,t} + \upsilon_4\hat{C}_t + (1 - \upsilon_4)(\hat{C}^*_t + \theta\hat{Q}_t)$$

$$\hat{p}_{H,t} = -(1 - \alpha)p_F^{1-\theta}\hat{T}_t$$

$$\hat{p}_{F,t} = \alpha p_H^{1-\theta}\hat{T}_t$$

$$\pi_{H,t} - \bar{\pi} = \phi[\eta\hat{Y}_{H,t} + \rho\hat{C}_t - \hat{p}_{H,t}] + \beta E_t(\pi_{H,t+1} - \bar{\pi})$$

$$\pi^*_{F,t} - \bar{\pi}^* = \phi^*[\eta\hat{Y}^*_{F,t} + \rho\hat{C}^*_t - \hat{p}_{F,t} + \hat{Q}_t] + \beta E_t\left(\pi^*_{F,t+1} - \bar{\pi}^*\right)$$

$$\hat{T}_t = \hat{T}_{t-1} + (\pi^*_{F,t} - \bar{\pi}^*) - (\pi_{H,t} - \bar{\pi}) + \Delta\hat{S}_t$$

$$\begin{aligned}\hat{Q}_t &= (1 - \alpha)p_H^{1-\theta}Q^{\theta-1}\hat{p}_{H,t} + \alpha p_F^{1-\theta}Q^{\theta-1}\hat{p}_{F,t} \\ &= p_H^{1-\theta}p_F^{1-\theta}Q^{\theta-1}(2\alpha - 1)\hat{T}_t\end{aligned}$$

$$\pi_t - \bar{\pi} = \alpha p_H^{1-\theta}(\pi_{H,t} - \bar{\pi}) + (1 - \alpha)p_F^{1-\theta}[(\pi^*_{F,t} - \bar{\pi}^*) + \Delta\hat{S}_t]$$

$$\pi^*_t - \bar{\pi} = \pi_t - \bar{\pi} + \Delta\hat{Q}_t - \Delta\hat{S}_t$$

where $\phi \equiv \tau/\kappa$, $\phi^* \equiv \tau/\kappa^*$ while these parameters are evaluated at the initial steady-state

$$\upsilon_1 = \frac{p_H Y_H}{C}$$

$$\upsilon_2 = \frac{k}{\Pi C}$$

$$\upsilon_3 = \frac{(1-\alpha)C}{(1-\alpha)C + \alpha C^* Q^\theta}$$

$$\upsilon_4 = \frac{\alpha C}{\alpha C + (1-\alpha)C^* Q^\theta}$$

$$\varpi_1 \equiv (1+i)\psi_d(1)k$$

$$\varpi_2 \equiv \frac{\chi_d(1)}{C}.$$

where we define $\psi_d(1)$ and $\chi_d(.)$ as the partial derivatives of $\chi(d_t/k_t)$ and $\psi(d_t/k_t)$ with respect to d.[1]

Note that under the assumption $\varpi_1 = \varpi_2/\beta\upsilon_2$ we can re-write the Euler equation and the budget constraint of the Home country in the following ways

$$E_t\hat{C}_{t+1} = \hat{C}_t + \rho^{-1}[\hat{\imath}_t^b - E_t(\pi_{t+1} - \bar{\pi})]$$

$$\hat{C}_t = \upsilon_1[\hat{p}_{H,t} + \hat{Y}_{H,t}] - \upsilon_2[\beta\hat{\imath}_t^b - (\pi_t - \bar{\pi})] + \upsilon_2\beta\hat{d}_t - \upsilon_2\hat{d}_{t-1}$$

where the effective borrowing rate $\hat{\imath}_t^b$ is defined as

$$\hat{\imath}_t^b - \hat{\imath}_t = \frac{\varpi_2}{\beta\upsilon_2}(\hat{d}_t - \hat{k}_t) = \varpi_1(\hat{d}_t - \hat{k}_t).$$

We maintain this assumption when calibrating the model, as explained in the text.

Optimal policy solves the maximization of (13) under the above-defined constraints, taking into account the two zero-lower-bound constraints. The equilibrium conditions of the optimal policy problem can be written in the general form (4) and therefore similar steps to those described in that section are used to solve for the response of the endogenous variables to the deleveraging shocks.

Note that by using the above restrictions, we can further write the second-order approximation of the utility as

$$\begin{aligned} U_t \;=\; & \bar{U} + \xi\bar{C}^{1-\rho}E_t\left\{\sum_{t=0}^{\infty}\beta^t\left[-\rho\frac{(\hat{C}_t - c)^2}{2} - \rho\frac{\bar{C}^*\bar{Q}}{\bar{C}}\frac{(\hat{C}_t^* - c^*)^2}{2} - \eta\frac{\bar{p}_H\bar{Y}_H}{\bar{C}}\frac{(\hat{Y}_{H,t} - y_H)^2}{2}\right.\right. \\ & -\eta\frac{\bar{p}_F\bar{Y}_F^*}{\bar{C}}\frac{(\hat{Y}_{F,t}^* - y_F^*)^2}{2} - \theta\bar{p}_H^{1-\theta}\bar{p}_F^{1-\theta}\alpha(1-\alpha)\left(1 + \frac{\bar{C}^*\bar{Q}}{\bar{C}}\frac{1}{\bar{Q}^{2(1-\theta)}}\right)\frac{(\hat{T}_t - s)^2}{2} \\ & \left.-\kappa\frac{\bar{p}_H\bar{Y}_H}{\bar{C}}\frac{(\pi_{H,t} - \bar{\pi})^2}{2} - \kappa^*\frac{\bar{p}_F\bar{Y}_F^*}{\bar{C}}\frac{(\pi_{F,t}^* - \bar{\pi}^*)^2}{2}\right] + \mathcal{O}(\|\cdot\|^3) \end{aligned} \qquad (14)$$

[1]The function $\chi(d_t/k_t)$ has been defined in Appendix B.

5. Model with deleveraging on foreign debt

In this section, we discuss the extension of the model to the case in which debt of the deleveraging country is denominated in foreign currency. In this case, the flow budget can be written as

$$P_t C_t = \int_0^1 W_t(j) L_t(j) dj + \Pi_t + \frac{S_t D_t}{1+i_t^*} - S_t D_{t-1} - f_t P_t \cdot \tilde{\chi}\left(\frac{S_t D_t}{P_t}\frac{1}{f_t}, \frac{S_t \bar{D}_t}{P_t}\frac{1}{f_t}\right) \qquad (15)$$

where now the function capturing the adjustment costs of changing the debt position has arguments expressed in terms of individual and aggregate real debt, in units of the domestic price index, with respect to a threshold f_t.

The following equilibrium conditions characterize now the consumers' problems in the Home country:

$$(C_t)^{-\rho}\left\{1 - (1+i_t^*)\psi\left(\frac{d_t^*}{f_t}\right)\right\} = \beta(1+i_t^*)E_t\left\{(C_{t+1})^{-\rho}\frac{P_t}{P_{t+1}}\frac{S_{t+1}}{S_t}\right\},$$

$$C_t = \frac{P_{H,t}Y_{H,t}}{P_t} + \frac{d_t^*}{(1+i_t^*)} - \frac{d_{t-1}^*}{\Pi_t}\frac{S_t}{S_{t-1}} - f_t\chi\left(\frac{d_t^*}{f_t}\right)$$

where we have defined $d_t^* = S_t D_t / P_t$ and

$$(C_t)^{-\rho} = \beta E_t\left\{(C_{t+1})^{-\rho}\frac{(1+i_t)}{\Pi_{t+1}}\right\},$$

since we are allowing for trading, within country H, of a risk-less bond denominated in domestic currency.

Note that in the final steady state now

$$\bar{C} = \bar{p}_H \bar{Y}_H - (1-\beta)\bar{\Pi}^{*-1}\bar{f},$$

$$\bar{Q}\bar{C}^* = \bar{p}_F \bar{Y}_F^* + (1-\beta)\bar{\Pi}^{*-1}\bar{f},$$

Finally the model equilibrium conditions in a first-order approximation are now

$$E_t\hat{C}_{t+1}^* = \hat{C}_t^* + \rho^{-1}[\hat{\imath}_t^* - E_t(\pi_{t+1}^* - \bar{\pi}^*)]$$

$$E_t\hat{C}_{t+1} = \hat{C}_t + \rho^{-1}[\hat{\imath}_t - E_t(\pi_{t+1} - \bar{\pi}_t)]$$

$$E_t\hat{C}_{t+1} = \hat{C}_t + \rho^{-1}[\hat{\imath}_t^* - E_t(\pi_{t+1} - \bar{\pi}) + E_t\Delta\hat{S}_{t+1} + \tilde{\varpi}_1(\hat{d}_t^* - \hat{f}_t)]$$

$$\hat{C}_t = \upsilon_1[\hat{p}_{H,t} + \hat{Y}_{H,t}] - \tilde{\upsilon}_2[\beta\hat{\imath}_t^* - (\pi_t - \bar{\pi}) + \Delta\hat{S}_t] + \tilde{\upsilon}_2\beta\hat{d}_t^* - \tilde{\upsilon}_2\hat{d}_{t-1}^* - \tilde{\varpi}_2(\hat{d}_t^* - \hat{f}_t)$$

$$\hat{Y}_{F,t}^* = -\theta\hat{p}_{F,t} + \upsilon_3\hat{C}_t + (1-\upsilon_3)(\hat{C}_t^* + \theta\hat{Q}_t)$$

$$\hat{Y}_{H,t} = -\theta \hat{p}_{H,t} + \upsilon_4 \hat{C}_t + (1-\upsilon_4)(\hat{C}_t^* + \theta \hat{Q}_t)$$

$$\hat{p}_{H,t} = -(1-\alpha) p_F^{1-\theta} \hat{T}_t$$

$$\hat{p}_{F,t} = \alpha p_H^{1-\theta} \hat{T}_t$$

$$\pi_{H,t} - \bar{\pi} = \phi[\eta \hat{Y}_{H,t} + \rho \hat{C}_t - \hat{p}_{H,t}] + \beta E_t(\pi_{H,t+1} - \bar{\pi})$$

$$\pi_{F,t}^* - \bar{\pi}^* = \phi^*[\eta \hat{Y}_{F,t}^* + \rho \hat{C}_t^* - \hat{p}_{F,t} + \hat{Q}_t] + \beta E_t \left(\pi_{F,t+1}^* - \bar{\pi}^*\right)$$

$$\hat{T}_t = \hat{T}_{t-1} + (\pi_{F,t}^* - \bar{\pi}^*) - (\pi_{H,t} - \bar{\pi}) + \Delta \hat{S}_t$$

$$\begin{aligned} \hat{Q}_t &= (1-\alpha) p_H^{1-\theta} Q^{\theta-1} \hat{p}_{H,t} + \alpha p_F^{1-\theta} Q^{\theta-1} \hat{p}_{F,t} \\ &= p_H^{1-\theta} p_F^{1-\theta} Q^{\theta-1} (2\alpha - 1) \hat{T}_t \end{aligned}$$

$$\pi_t - \bar{\pi} = \alpha p_H^{1-\theta}(\pi_{H,t} - \bar{\pi}) + (1-\alpha) p_F^{1-\theta}[(\pi_{F,t}^* - \bar{\pi}^*) + \Delta \hat{S}_t]$$

$$\pi_t^* - \bar{\pi} = \pi_t - \bar{\pi} + \Delta \hat{Q}_t - \Delta \hat{S}_t$$

where now

$$\tilde{\upsilon}_2 = \frac{f}{\Pi^* C}$$

$$\tilde{\varpi}_1 \equiv (1+i^*) \psi_{d^*}(1) f$$

$$\tilde{\varpi}_2 \equiv \chi_{d^*}(1) f / C.$$