Appendix to "Optimal Monetary Policy in a Currency Area"

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Appendix A

In this appendix we show that assets and consumption are stationary variables.

Lemma 1 Given equation (10) in the text and the optimality conditions describing the optimal path of consumption (i),(ii),(iii), and given the initial condition $B_{t-1}^{i}=0$ for each $i=H$, F it follows that $B_{T}^{i}=0$ $\forall T\geq t$ for each $i=H$, F.

Proof. We use a proof by induction where the proposition $\pounds_{T}$ has been defined as $\pounds_{T}:=B_{T}^{i}=0$ for each $i=H,F$ and $\forall T\geq t-1$.

By using the assumption that $B_{t-1}^{i}=0$ for each i, we have that $\pounds_{t-1}=0$. It remains to prove that for a $T>t$ if $\pounds_{T-1}$ is true, $\pounds_{T}$ is also true. If $\pounds_{T-1}$ is true, $B_{T-1}^{i}=0$ for each i. By using (10) we have that in a generic state $s_{T}\in S_{T}$ at date T

$$\begin{aligned} C_{T}^{H} &= C_{T}^{W}-\frac{B_{T}^{H}}{P_{T}(1+R_{T})}, \\ C_{T}^{F} &= C_{T}^{W}-\frac{B_{T}^{F}}{P_{T}(1+R_{T})}. \end{aligned}$$

At date $T+1$ in each state $s_{T+1}\in S_{T+1}$ the optimality conditions of the representative household i for each $i=H$ and F are

$$U_{C}(C_{T+1}^{i}(s_{T+1}))=(1+R_{T+1})\beta\mathrm{E}_{T+1}\left\{U_{C}(C_{T+2}^{i})\frac{P_{T+1}}{P_{T+2}}\right\}, \tag{A.1}$$

$$\{U_{C}(C_{T+s}^{i})\}=\mathrm{E}_{T+s}\left\{(1+R_{T+s})\beta U_{C}(C_{T+s+1}^{i})\frac{P_{T+s}}{P_{T+s+1}}\right\}\text{ for }s>1, \tag{A.2}$$

$$\left\{\sum_{k=T+1}^{\infty}R_{T+1,k}^{r}C_{k}^{i}(s_{k})\right\}=\frac{B_{T}^{i}}{P_{T+1}}+\left\{\sum_{k=T+1}^{\infty}R_{T+1,k}^{r}C_{k}^{W}(s_{k})\right\}, \tag{A.3}$$

where the discount factor has been defined as

$$\begin{aligned} R^r_{T+1,k} &= \frac{P_k}{P_{T+1}\prod_{s=T+1}^{k-1}(1+R_s)} \text{ for } k > T+1, \\ R^r_{T+1,T+1} &= 1. \end{aligned}$$

and where (A.3) has been obtained after iterating (10) and it consists of a set of conditions corresponding to any possible history starting from each state $s_{T+1} \in S_{T+1}$ at date $t+1$. If $B^H_T = B^F_T$ the optimal allocations of consumptions of households of region H and F are exactly the same looking ahead from period $T+1$ in each state $s_{T+1} \in S_{T+1}$. Indeed, the expected discounted value of human wealth (the expression in the curly brackets of the RHS of (A.3)) is the same across regions and is taken as given by the households in their consumption decisions. Thus we can write $C^H_{T+1}(s_{T+1})$ and $C^F_{T+1}(s_{T+1})$ as they were implicitly defined by the same indirect function, which is state-dependent, of respectively B^H_T and B^F_T,

$$\begin{aligned} C^H_{T+1}(s_{T+1}) &= \Gamma_{s_{T+1}}(B^H_T), \\ C^F_{T+1}(s_{T+1}) &= \Gamma_{s_{T+1}}(B^F_T), \end{aligned}$$

and this is true for each $s_{T+1} \in S_{T+1}$. Moreover $\Gamma_{s_{T+1}}$ is a non-decreasing monotone function of the initial level of assets, B^H_T or B^F_T. From the equilibrium condition at time T we have $nB^H_T + (1-n)B^F_T = 0$. If $B^H_T > 0$, it follows that $C^H_{T+1}(s_{T+1}) > C^F_{T+1}(s_{T+1})$ for each $s_{T+1} \in S_{T+1}$, while $C^H_T < C^F_T$. But this violates the optimality condition[1]

$$\mathrm{E}_T \left\{ \left[\frac{U_C(C^H_{T+1})}{U_C(C^H_T)} - \frac{U_C(C^F_{T+1})}{U_C(C^F_T)} \right] \frac{P_T}{P_{T+1}} \right\} = 0, \tag{A.4}$$

because the term in the square bracket is negative across all the states and prices are always positive. It should be then that $B^H_T = B^F_T = 0$. ■

A corollary of this conclusion is that there is perfect risk sharing of consumption between regions, i.e. $C^H = C^F = C^W$ at any time and at any state.

Appendix B

This appendix contains the proofs of some propositions of section 3.of the paper.

Proposition 1

Proof. We show here that terms of trade is independent of monetary policy when $\alpha^H = \alpha^F$. In this case $k^H_C = k^F_C$ and $k^H_T = k^F_T = k_T$ By taking the difference of equations (22) and (21), we obtain that

$$\pi^R_t = -k_T(\widehat{T}_t - \widetilde{T}_t) + \beta \mathrm{E}_t \pi^R_{t+1}, \tag{B.1}$$

where the pressure on relative inflation is given by the deviations of the terms of trade from their natural rate. Noting that $\pi^R_t = \widehat{T}_t - \widehat{T}_{t-1}$ we obtain

$$\mathrm{E}_t\widehat{T}_{t+1} - \frac{1+\beta+k_T}{\beta}\widehat{T}_t + \frac{1}{\beta}\widehat{T}_{t-1} = -\frac{k_T}{\beta}\widetilde{T}_t. \tag{B.2}$$

[1] I am grateful to Cedric Tille for pointing out this last observation

where the stochastic difference equation (B.2) has always one eigenvalue with modulus less than 1 and one which is bigger than $1/\beta$. The unique and stable solution is given by

$$\widehat{T}_t = \lambda_1 \widehat{T}_{t-1} + \lambda_1 k_T \mathrm{E}_t \sum_{s=t}^{\infty} \left(\frac{1}{\lambda_2}\right)^{s-t} \widetilde{T}_s,$$

where λ_1 is a positive eigenvalue of the second order difference equation (B.2), with λ_1 less than 1. ■

Proposition 2

Proof. We have that the AS equation of the region F, in the case its prices are sticky, is

$$\pi_t^F = -k_T^F [n\widehat{T}_t - \frac{k_C^F}{k_T^F}(\widehat{C}_t^W - \widetilde{C}_{F,t})] + \beta \mathrm{E}_t \pi_{t+1}^F, \tag{B.3}$$

where we have that

$$\widetilde{C}_{F,t} \equiv \frac{\eta}{\rho+\eta}(\overline{Y}_t^F - g_t^F).$$

Under the assumption that prices in region F are flexible, we can observe that the term of trade is implied by the term in the square brackets of (B.3), namely

$$n\widehat{T}_t = \frac{k_C^F}{k_T^F}(\widehat{C}_t^W - \widetilde{C}_{F,t}). \tag{B.4}$$

Similarly rearranging the AS equation of region H we obtain

$$\pi_t^H = k_T^H [(1-n)\widehat{T}_t + \frac{k_C^H}{k_T^H}(\widehat{C}_t^W - \widetilde{C}_{H,t})] + \beta \mathrm{E}_t \pi_{t+1}^H, \tag{B.5}$$

where we have that

$$\widetilde{C}_{H,t} \equiv \frac{\eta}{\rho+\eta}(\overline{Y}_t^H - g_t^H).$$

After plugging (B.4) into (B.5) we obtain

$$\pi_t^H = \frac{k_C}{n}[\widehat{C}_t^W - n\widetilde{C}_{H,t} - (1-n)\widetilde{C}_{F,t}] + \beta \mathrm{E}_t \pi_{t+1}^H.$$

By noting that $\widetilde{C}_t^W = n\widetilde{C}_{H,t} + (1-n)\widetilde{C}_{F,t}$, we reach the conclusion that by stabilizing the inflation rate in region H at all date t monetary authority reaches a path of consumption consistent with its efficient level at all dates t. Moreover if $\widehat{C}_t = \widetilde{C}_t$ it follows from (B.4) that $\widehat{T}_t = \widetilde{T}_t$. ■

Proposition 3

Proof. We recall equations (21) and (22)

$$\pi_t^H = (1-n)k_T^H(\widehat{T}_t - \widetilde{T}_t) + k_C^H y_t^W + \beta E_t \pi_{t+1}^H, \tag{B.6}$$
$$\pi_t^F = -n k_T^F(\widehat{T}_t - \widetilde{T}_t) + k_C^F y_t^W + \beta E_t \pi_{t+1}^F. \tag{B.7}$$

By contradiction if $\widehat{T}_T = \widetilde{T}_T$ and $y_T^W = 0$ at all dates $T \geq t$, we have that $\pi_T^R = \pi_T^F - \pi_T^H = 0$ at all dates $T \geq t$, implying that $\widehat{T}_t = \widehat{T}_{t-1} = 0$ (given the initial condition $\widehat{T}_{t-1} = 0$), which contradicts $\widehat{T}_t = \widetilde{T}_t$ unless $\widetilde{T}_t = 0$ and this is for each date t. ■

Proposition 5

Proof. The optimal plan can be obtained by taking the first order condition of the following Lagrangian[2]

$$\begin{gathered}\mathrm{E}_0\{\textstyle\sum_{t=0}^{\infty}\beta^t\{\Lambda\cdot[y_t^W]^2+n(1-n)\Gamma\cdot[\widehat{T}_t-\widetilde{T}_t]^2+\gamma\cdot(\pi_t^H)^2+(1-\gamma)\cdot(\pi_t^F)^2+\\ +2n\phi_{1,t}[\pi_t^H-(1-n)k_T^H(\widehat{T}_t-\widetilde{T}_t)-k_C^H y_t^W-\beta\pi_{t+1}^H]+\\ +2(1-n)\phi_{2,t}[\pi_t^F+nk_T^F(\widehat{T}_t-\widetilde{T}_t)-k_C^F y_t^W-\beta\pi_{t+1}^F]\\ +2\phi_{3,t}[\widehat{T}_t-\widehat{T}_{t-1}-\pi_t^F+\pi_t^H]\},\end{gathered}$$

where $n\cdot\phi_{1,t}$ and $(1-n)\cdot\phi_{2,t}$ are the Lagrangian multipliers associated with the constraints (B.6) and (B.7), respectively; $\phi_{3,t}$ is the Lagrangian multiplier associated with the terms of trade identity[3]

$$\widehat{T}_t = \widehat{T}_{t-1} + \pi_t^F - \pi_t^H, \tag{B.8}$$

The first-order conditions are

$$\Lambda y_t^W - nk_C^H\phi_{1,t} - (1-n)k_C^F\phi_{2,t} = 0, \tag{B.9}$$

$$\gamma\pi_t^H + n(\phi_{1,t} - \phi_{1,t-1}) + \phi_{3,t} = 0, \tag{B.10}$$

$$(1-\gamma)\pi_t^F + (1-n)(\phi_{2,t} - \phi_{2,t-1}) - \phi_{3,t} = 0, \tag{B.11}$$

$$n(1-n)\Gamma(\widehat{T}_t-\widetilde{T}_t)-n(1-n)k_T^H\phi_{1,t}+n(1-n)k_T^F\phi_{2,t}+\phi_{3,t}-\beta\phi_{3,t+1}=0, \tag{B.12}$$

obtained by minimizing the Lagrangian with respect to y_t^W, π_t^H, π_t^F, $\widehat{T}_t$. These conditions hold at each date t with $t \geq 1$. They also hold at time 0, given the initial conditions on the absence of commitment at time 0

$$\phi_{1,-1} = \phi_{2,-1} = 0,$$

and the initial condition on $\widehat{T}_{-1}$ which is imposed to be equal to zero.[4] The optimal bounded plan is a set of bounded processes $\{y_t^W, \pi_t^H, \pi_t^F, \widehat{T}_t, \phi_{1,t}, \phi_{2,t}, \phi_{3,t}\}$ that

[2] Note that we have omitted the term Ω, by normalizing the Lagrange multiplier. We have also multiplied the Lagrange multiplier by a factor of two in order to eliminate a recurrent factor of two from the first-order conditions.

[3] The constraint (19) in the text is not relevant in the optimization problem, since there is no cost due to the volatility of the nominal interest rate in the loss function. Given the optimal path for the sequences $\{y_t^W, \widehat{T}_t, \pi_t^H, \pi_t^F\}_{t=0}^{+\infty}$, the nominal interest rate is adjusted residually following (19). Thus, it follows that the optimal allocation of $\{y_t^W, \widehat{T}_t, \pi_t^H, \pi_t^F\}_{t=0}^{+\infty}$ will be independent of the path of the natural rate of interest, $\{\widehat{R}\}_{t=0}^{+\infty}$.

[4] This optimal plan is not time-consistent. Time consistency would have required that the Lagrange multiplier were zero at all dates, but this is not feasible as a solution.

satisfy conditions (B.6), (B.7), (B.8), (B.9), (B.10), (B.11) and (B.12), given the initial conditions and given the process for $\widetilde{T}_t$.[5] Noting that each of the first-order conditions hold at each date t, they should hold under commitment also conditional upon the information set at each date t. We can rearrange the conditions characterizing the optimal plan as

$$QE_t x_{t+1} = M x_t + v\widetilde{T}_t, \tag{B.13}$$

where Q and M are 9×9 matrices, $x_t^{'} \equiv [y_t^W, \pi_t^H, \pi_t^F, \phi_{3,t}, \phi_{1,t}, \phi_{2,t}, \widehat{T}_{t-1},\ s_t, w_t]$ where we have defined $s_t \equiv \phi_{1,t-1},\ w_t \equiv \phi_{2,t-1}$, and v is a 9×1 vector. Considering a bounded stochastic process for the shock $\widetilde{T}_t$, a bounded optimal plan exist and it is unique if and only if the matrix $(Q)^{-1}M$ has exactly three eigenvalues inside the unit circle; in fact in the system of stochastic difference equations (B.13) there are three predetermined variables. If we assume an autoregressive process for the shock $\widetilde{T}_t = \phi\widetilde{T}_{t-1} + \varepsilon_t$, where ε_t is a white noise and $0 \leq \phi < 1$, the unique bounded solution can be written as

$$\begin{aligned} p_t &= Lz_t + l\widetilde{T}_t, \\ z_t &= Hz_{t-1} + h\widetilde{T}_t, \\ \widetilde{T}_t &= \phi\widetilde{T}_{t-1} + \varepsilon_t \end{aligned}$$

where $p_t' \equiv [y_t^W, \pi_t^H, \pi_t^F, \phi_{3,t}]$ $z_t' \equiv [\widehat{T}_t, \phi_{1,t}, \phi_{2,t}]$, L is a 4×4 matrix, H is a 3×3 matrix while l and h are 4×1 and 3×1 vectors, respectively. Furthermore we can write the solution of the terms of trade as

$$\begin{aligned} \det[I - HL]\widehat{T}_t &= h_1\widetilde{T}_t + [H_{13}h_3 + H_{12}h_2 - H_{22}h_1 - H_{33}h_1]\widetilde{T}_{t-1} \\ &\quad +[H_{22}H_{33}h_1 - H_{23}H_{32}h_1 - H_{12}H_{33}h_2 + H_{12}H_{23}h_3 + \\ &\quad +H_{13}H_{32}h_2 - H_{13}H_{22}h_3]\widetilde{T}_{t-2}, \end{aligned}$$

where h_j are elements of the vector h, and H_{ij} are elements of the matrix H, or more compactly as

$$G(L)\widehat{T}_t = V(L)\widetilde{T}_t, \tag{B.14}$$

where L is the lag operator and $G(L)$ and $V(L)$ are polynomials in the lag operators respectively of the third order and of the second order. Furthermore it can be shown that also y_t^W, π_t^H, π_t^F have the same representations as in (B.14) with different polynomials but of the same orders ■

Appendix C

In this appendix, we derive the log-linear approximation of region H's AS equation, equation (21) in the text. The derivation of the region F's supply side follows in

[5] We are considering only bounded solutions, thus we can neglect the set of transversality conditions.

a specular way. Given the sequences $\{C_t\}$, the sequences of shocks and the initial conditions, the optimal paths of prices $\{\widetilde{p}_t(h), P_{H,t}\}$ is described by the following conditions

$$\widetilde{p}_t(h) = \frac{\sigma}{(\sigma-1)(1-\tau^H)} \frac{\mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k V_y(\widetilde{y}^d_{t,t+k}(h), z^H_{t+k}) \widetilde{y}^d_{t,t+k}(h)}{\mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k \lambda_{t+k} \widetilde{y}^d_{t,t+k}(h)}, \quad \text{(D.15)}$$

$$P^{1-\sigma}_{H,t} = \alpha^H P^{1-\sigma}_{H,t-1} + (1-\alpha^H) \widetilde{p}_t(h)^{1-\sigma}, \quad \text{(D.16)}$$

where

$$\widetilde{y}^d_{t,t+k}(h) = \left(\frac{\widetilde{p}_t(h)}{P_{H,t+k}} \right)^{-\sigma} [T^{1-n}_{t+k} C_{t+k} + G^H_{t+k}]. \quad \text{(D.17)}$$

We can write (D.15) as

$$0 = \mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k \{[(1-\sigma)(1-\tau^H)\lambda_{t+k} \widetilde{p}_t(h) +$$
$$+ \sigma V_y(\widetilde{y}^d_{t,t+k}(h), z^H_{t+k})] \widetilde{y}^d_{t,t+k}(h)\},$$

and after substituting the expression for λ_{t+k}

$$\mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k \left\{ \left[\begin{array}{c} (1-\sigma)(1-\tau^H) U_C(C_{t+k}) \frac{\widetilde{p}_t(h)}{P_{t+k}} + \\ + \sigma V_y(\widetilde{y}^d_{t,t+k}(h), z^H_{t+k}) \end{array} \right] \right\} \widetilde{y}^d_{t,t+k}(h) = 0,$$

or

$$\mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k \left\{ \left[\begin{array}{c} (1-\sigma)(1-\tau^H) U_C(C_{t+k}) \frac{\widetilde{p}_t(h)}{P_{H,t+k}} T^{n-1}_{t+k} + \\ + \sigma V_y(\widetilde{y}^d_{t,t+k}(h), z^H_{t+k}) \end{array} \right] \widetilde{y}^d_{t,t+k}(h) \right\} = 0, \quad \text{(D.18)}$$

where $T_{t+k} = P_{F,t+k}/P_{H,t+k}$. We take a log-linear approximation of this equilibrium condition around a steady state in which $C_t = \overline{C}$, $T_t = 1$, $\widetilde{p}_t(h)/P_{H,t} = 1$, $G^H_t = 0$, $z^H_t = 0$ and $(1-\tau^H)U_C(\overline{C}) = \frac{\sigma}{\sigma-1} V_y(\overline{C}, 0)$ at all times, obtaining

$$0 = \mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k \{(1-\sigma)(1-\tau^H) U_C(\overline{C}) \widehat{p}_{t,t+k} +$$
$$+ (1-\sigma)(1-\tau^H) U_C(\overline{C})[-(1-n)\widehat{T}_{t+k}]$$
$$+ (1-\sigma)(1-\tau^H) U_{CC}(\overline{C}) \overline{C} \widehat{C}_{t+k} + \sigma \overline{C} V_{yy}(\overline{C}, 0)[-\sigma \widehat{p}_{t,t+k} +$$
$$+ (1-n)\widehat{T}_{t+k} + \widehat{C}_{t+k} + g^H_{t+k}] + \sigma V_{yz}(\overline{C}, 0) \widehat{z}^H_{t+k}\}$$

where $\widehat{p}_{t,t+k} = \ln(\widetilde{p}_t(h)/P_{H,t+k})$. We can further simplify the equation above to

$$0 = \mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H \beta)^k \{(\widehat{p}_{t,t+k} - (1-n)\widehat{T}_{t+k} - \rho \widehat{C}_{t+k} - \eta[-\sigma \widehat{p}_{t,t+k} + (1-n)\widehat{T}_{t+k}$$
$$+ \widehat{C}_{t+k} + g^H_{t+k} - \overline{Y}^H_t]\},$$

where $\rho \equiv -U_{CC}(\overline{C})\overline{C}/U_C(\overline{C})$ and $\eta \equiv V_{yy}(\overline{C},0)\overline{C}/V_y(\overline{C},0)$, while we have define $\overline{Y}_t^H$ such that $V_{yz}(\overline{C},0)\widehat{z}_{t+k}^H \equiv -\overline{C}V_{yy}(\overline{C},0)\overline{Y}_t^H$. We note that

$$\widehat{p}_{t,t+k} = \widehat{p}_{t,t} - \sum_{s=1}^{k} \pi_{H,t+s}$$

we can then simplify to

$$\frac{\widehat{p}_{t,t}}{1-\alpha^H\beta} = \mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H\beta)^k [\frac{1+\eta}{1+\sigma\eta}(1-n)\widehat{T}_{t+k} + \frac{\rho+\eta}{1+\sigma\eta}\widehat{C}_{t+k}$$
$$+\frac{\eta}{1+\sigma\eta}(g_{t+k}^H - \overline{Y}_{t+k}^H)] + \mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H\beta)^k \left[\sum_{s=1}^{k} \pi_{H,t+s}\right]. \tag{D.19}$$

Log-linearizing (D.16), we obtain

$$\widehat{p}_{t,t} = \frac{\alpha^H}{1-\alpha^H}\pi_t^H$$

Thus we can simplify (D.19) further to

$$\frac{\pi_t^H}{1-\alpha^H\beta}\frac{\alpha^H}{1-\alpha^H} = \mathrm{E}_t \sum_{k=0}^{\infty} (\alpha^H\beta)^k [\frac{1+\eta}{1+\sigma\eta}(1-n)\widehat{T}_{t+k} + \frac{\rho+\eta}{1+\sigma\eta}\widehat{C}_{t+k} +$$
$$+\frac{\eta}{1+\sigma\eta}(g_{t+k}^H - \overline{Y}_{t+k}^H)] + \mathrm{E}_t \sum_{k=1}^{\infty} (\alpha^H\beta)^k \frac{\pi_{t+k}^H}{1-\alpha\beta}$$

We obtain

$$\begin{aligned} \pi_t^H \quad = \quad & (1-\alpha^H\beta)\frac{1-\alpha^H}{\alpha^H}\frac{1+\eta}{1+\sigma\eta}(1-n)\widehat{T}_t + (1-\alpha^H\beta)\frac{1-\alpha^H}{\alpha^H}\frac{\rho+\eta}{1+\sigma\eta}\widehat{C}_t \\ & +(1-\alpha^H\beta)\frac{1-\alpha^H}{\alpha^H}\frac{\eta}{1+\sigma\eta}(g_t - \overline{Y}_t) + \beta\mathrm{E}_t\pi_{t+1}^H \end{aligned} \tag{D.20}$$

noting that the natural rate of world consumption and of the terms of trade, which will arise when prices are flexible, are defined as

$$\begin{aligned} \widetilde{C}_t \quad & \equiv \quad \frac{\eta}{\rho+\eta}(\overline{Y}_t^W - g_t^W), \\ \widetilde{T}_t \quad & \equiv \quad \frac{\eta}{1+\eta}(g_t^R - \overline{Y}_t^R). \end{aligned}$$

we can simplify the equation above to

$$\pi_t^H = (1-n)k_T^H(\widehat{T}_t - \widetilde{T}_t) + k_C^H(\widehat{C}_t - \widetilde{C}_t) + \beta E_t\pi_{t+1}^H, \tag{D.21}$$

which corresponds to equation (21) in the text (note that $\widehat{C}_t = \widehat{C}_t^W$) where

$$
\begin{aligned}
k_T^H &\equiv (1 - \alpha^H \beta) \frac{1 - \alpha^H}{\alpha^H} \frac{1 + \eta}{1 + \sigma\eta} \\
k_C^H &\equiv (1 - \alpha^H \beta) \frac{1 - \alpha^H}{\alpha^H} \frac{\rho + \eta}{1 + \sigma\eta}.
\end{aligned}
$$

Appendix D

In this appendix we derive the utility-based loss function, equation (26) in the text. We follow Rotemberg and Woodford (1997,1998) and Woodford (1999a). The average utility flow among all the households belonging to region H is

$$w_t^H = U(C_t) - \frac{\int_0^n v(y_t(h), z_t^H)dh}{n}, \tag{E.1}$$

while that of region F is

$$w_t^F = U(C_t) - \frac{\int_{1-n}^1 v(y_t(f), z_t^F)df}{1-n}. \tag{E.2}$$

The welfare criterion of the Central Bank in the currency area is the discounted value of a weighted average of the average utility flows of the regions,

$$W = \mathrm{E}_0 \sum_{j=0}^{\infty} \beta^j (nw_{t+j}^H + (1-n)w_{t+j}^F). \tag{E.3}$$

We take a Taylor expansion of each term of the utility function. Taking a second-order linear expansion of $U(C_t)$ around the steady state value $\overline{C}$ defined by equation (A.22), we obtain

$$U(C_t) = U(\overline{C}) + U_C(C_t - \overline{C}) + \frac{1}{2}U_{CC}(C_t - \overline{C})^2 + o(\|\xi\|^3), \tag{E.4}$$

where in $o(\|\xi\|^3)$ we group all the terms that are of third or higher order in the deviations of the various variables from their steady-state values. Furthermore expanding C_t with a second-order Taylor approximation we obtain

$$C_t = \overline{C}(1 + \widehat{C}_t + \frac{1}{2}\widehat{C}_t^2) + o(\|\xi\|^3), \tag{E.5}$$

where $\widehat{C}_t = \ln(C_t/\overline{C})$. Substituting (E.5) into (E.4) we obtain

$$U(C_t) = U_C\overline{C}\widehat{C}_t + \frac{1}{2}(U_C\overline{C} + U_{CC}\overline{C}^2)\widehat{C}_t^2 + \text{t.i.p.} + o(\|\xi\|^3), \tag{E.6}$$

which can be written as

$$U(C_t) = U_C\overline{C}[\widehat{C}_t + \frac{1}{2}(1-\rho)\widehat{C}_t^2] + \text{t.i.p.} + o(\|\xi\|^3),$$

where we have defined $\rho \equiv -U_{CC}\overline{C}/U_C$ and where in t.i.p. we include all the terms that are independent of monetary policy. Similarly we take a second-order Taylor expansion of $v(y_t(h), z_t^H)$ around a steady state where $y_t(h) = \overline{Y}^H$ for each h, and at each date t, and where $z_t^H = 0$ at each date t. We obtain

$$v(y_t(h), z_t^H) = v(\overline{Y}^H, 0) + v_y(y_t(h) - \overline{Y}^H) + v_z z_t^H + \frac{1}{2} v_{yy}(y_t(h) - \overline{Y}^H)^2$$
$$+ v_{yz}(y_t(h) - \overline{Y}^H) z_t^H + \frac{1}{2} v_{zz}(z_t^H)^2 + o(\|\xi\|^3), \qquad \text{(E.7)}$$

where $\widehat{y}_t(h) = \ln(y_t(h)/\overline{Y}^H)$. Here we recall that

$$y(h) = \left(\frac{p(h)}{P_H}\right)^{-\sigma} \left[(T)^{1-n} C^W + G^H\right],$$

which can be rewritten as

$$y(h) = y^d(h) + y^g(h),$$

where we have defined

$$y^d(h) \equiv \left(\frac{p(h)}{P_H}\right)^{-\sigma} (T)^{1-n} C^W,$$
$$y^g(h) \equiv \left(\frac{p(h)}{P_H}\right)^{-\sigma} G^H.$$

Here we take a second order Taylor expansion of $y_t^d(h)$ and $y_t^g(h)$ obtaining

$$y_t^d(h) = \overline{Y}^H \cdot (1 + \widehat{y}_t^d(h) + \frac{1}{2} \cdot [\widehat{y}_t^d(h)]^2) + o(\|\xi\|^3),$$
$$y_t^g(h) = \overline{Y}^H \cdot (\widehat{y}_t^g(h) + \frac{1}{2} \cdot [\widehat{y}_t^g(h)]^2) + o(\|\xi\|^3).$$

We note that $y_t^g(h)$ can be neglected because in its expansion, the term of order less than $o(\|\xi\|^3)$ are independent of monetary policy, being the shock G^H equal to zero in the steady state. We can simplify (E.7) to

$$v(y_t(h), z_t^H) = v_y \overline{Y}^H \cdot [\widehat{y}_t^d(h) + \frac{1}{2} \cdot \widehat{y}_t^d(h)^2 + \frac{\eta}{2} \cdot \widehat{y}_t(h)^2$$
$$- \eta \cdot \widehat{y}_t(h) \overline{Y}_t^H] + \text{ t.i.p.} + o(\|\xi\|^3), \qquad \text{(E.8)}$$

where $\overline{Y}_t^H$ has been defined by the relation $v_{yz} z_t^H \equiv -v_{yy} \overline{Y}^H \overline{Y}_t^H$ and we have that $\eta \equiv V_{yy}(\overline{Y}^H, 0)\overline{Y}^H / V_y(\overline{Y}^H, 0)$. Our steady state with zero inflation implies the following conditions, respectively for region H

$$(1 - \tau^H) U_C(\overline{C}) = \frac{\sigma}{\sigma - 1} \overline{T}^{1-n} V_y\left(\overline{T}^{1-n} \overline{C}, 0\right), \qquad \text{(E.9)}$$

and for region F

$$(1 - \tau^F) U_C(\overline{C}) = \frac{\sigma}{\sigma - 1} \overline{T}^{-n} V_y\left(\overline{T}^{-n} \overline{C}, 0\right), \qquad \text{(E.10)}$$

which can be rewritten as

$$(1-\Phi^H)U_C(\overline{C}) = \overline{T}^{1-n}V_y\left(\overline{T}^{1-n}\overline{C},0\right), \tag{E.11}$$

$$(1-\Phi^F)U_C(\overline{C}) = \overline{T}^{-n}V_y\left(\overline{T}^{-n}\overline{C},0\right), \tag{E.12}$$

after having defined

$$\begin{aligned}
(1-\Phi^H) &\equiv (1-\tau^H)\frac{\sigma-1}{\sigma}, \\
(1-\Phi^F) &\equiv (1-\tau^F)\frac{\sigma-1}{\sigma}.
\end{aligned}$$

In the efficient equilibrium, we have that $\Phi^H = \Phi^F = 0$. As outlined in Woodford (1999a), we have to restrict our attention on steady state in which the deviations of Φ^H and Φ^F are of order at least $o(\|\xi\|)$. We also restrict the analysis to the case in which $\Phi^H = \Phi^F$. In this case we have that $\overline{Y}^H = \overline{Y}^F = \overline{C}$. In the neighbor of the efficient level of production and consumption we can write the steady state term of trade and consumption, by using conditions (E.11) and (E.12), as

$$\begin{aligned}
\overline{T} &= 1, \\
\ln\overline{C}/C^* &= -\frac{n\Phi^H + (1-n)\Phi^F}{\rho+\eta},
\end{aligned} \tag{E.13}$$

where C^* is the efficient level of consumption. By using (E.11) we can write (E.8) as

$$\begin{aligned}
v(y_t(h), z_t^H) &= U_C\overline{C}\cdot[(1-\Phi)\cdot\widehat{y}_t^d(h) + \frac{1}{2}\cdot\widehat{y}_t^d(h)^2 + \frac{\eta}{2}\cdot\widehat{y}_t(h)^2 \\
&\quad -\eta\cdot\widehat{y}_t(h)\overline{Y}_t^H] + \text{ t.i.p.} + o(\|\xi\|^3).
\end{aligned} \tag{E.14}$$

Here we integrate (E.14) across the households belonging to region H, obtaining

$$\begin{aligned}
\frac{\int_0^n v(y_t(h), z_t^H)dh}{n} &= U_C\overline{C}\cdot\{(1-\Phi)\cdot E_h\widehat{y}_t^d(h) + \frac{1}{2}\cdot[\text{var}_h\widehat{y}_t^d(h) + [E_h\widehat{y}_t^d(h)]^2] \\
&\quad +\frac{\eta}{2}\cdot[\text{var}_h\widehat{y}_t(h) + [E_h\widehat{y}_t(h)]^2] - \eta E_h\widehat{y}_t(h)\overline{Y}_t^H\} \\
&\quad +\text{t.i.p.} + o(\|\xi\|^3).
\end{aligned} \tag{E.15}$$

Using the aggregator (**??**) we can write

$$Y_{H,t} = Y_{H,t}^d + Y_{H,t}^g,$$

where

$$\begin{aligned}
Y_{H,t}^d &= T_t^{1-n}C_t^W, \\
Y_{H,t}^g &= G^H.
\end{aligned}$$

We take a second-order approximation of the aggregators obtaining

$$\widehat{Y}_{H,t} = E_h \widehat{y}_t(h) + \frac{1}{2}\left(\frac{\sigma-1}{\sigma}\right) \mathrm{var}_h \widehat{y}_t(h) + o(\|\xi\|^3),$$
$$\widehat{Y}^d_{H,t} = E_h \widehat{y}^d_t(h) + \frac{1}{2}\left(\frac{\sigma-1}{\sigma}\right) \mathrm{var}_h \widehat{y}^d_t(h) + o(\|\xi\|^3). \qquad \text{(E.16)}$$

Finally substituting (E.16) into (E.15) we obtain

$$\begin{aligned} \frac{\int_0^n v(y_t(h), z_t)}{n} &= U_C \overline{C} \cdot [(1-\Phi^H) \cdot \widehat{Y}^d_{H,t} + \frac{1}{2} \cdot [\widehat{Y}^d_{H,t}]^2 + \frac{\eta}{2} \cdot [\widehat{Y}_{H,t}]^2 \\ &\quad + \frac{1}{2}(\sigma^{-1}+\eta) \cdot \mathrm{var}_h \widehat{y}_t(h) - \eta \widehat{Y}^d_{H,t} \overline{Y}^H_t] \\ &\quad + \text{t.i.p.} + o(\|\xi\|^3) \end{aligned} \qquad \text{(E.17)}$$

where we have used the fact that $\mathrm{var}_h \widehat{y}_t(h) = \mathrm{var}_h \widehat{y}^d_t(h)$.

Combining (E.17) and (E.6) into (E.1), we obtain

$$\begin{aligned} w^H_t &= U_C \overline{C} [\widehat{C}_t + \frac{1}{2}(1-\rho)\widehat{C}^2_t - (1-\Phi^H) \cdot \widehat{Y}^d_{H,t} - \frac{1}{2} \cdot [\widehat{Y}^d_{H,t}]^2 - \frac{\eta}{2} \cdot [\widehat{Y}_{H,t}]^2 \\ &\quad - \frac{1}{2}(\sigma^{-1}+\eta) \cdot \mathrm{var}_h \widehat{y}_t(h) + \eta \widehat{Y}^d_{H,t} \overline{Y}^H_t] \\ &\quad + \text{t.i.p.} + o(\|\xi\|^3), \end{aligned} \qquad \text{(E.18)}$$

while for region F we have

$$\begin{aligned} w^F_t &= U_C \overline{C} [\widehat{C}_t + \frac{1}{2}(1-\rho)\widehat{C}^2_t - (1-\Phi^F) \cdot \widehat{Y}^d_{F,t} - \frac{1}{2} \cdot [\widehat{Y}^d_{F,t}]^2 - \frac{\eta}{2} \cdot [\widehat{Y}_{F,t}]^2 \\ &\quad - \frac{1}{2}(\sigma^{-1}+\eta) \cdot \mathrm{var}_f \widehat{y}_t(f) + \eta \widehat{Y}^d_{F,t} \overline{Y}^F_t] \\ &\quad + \text{t.i.p.} + o(\|\xi\|^3). \end{aligned} \qquad \text{(E.19)}$$

Taking a linear combination of (E.18) and (E.19) with weight n, we obtain

$$\begin{aligned} w_t &= U_C \overline{C} \{\widehat{C}_t \cdot [n\Phi^H + (1-n)\Phi^F] + \frac{1}{2}(1-\rho)\widehat{C}^2_t \\ &\quad - \frac{1}{2} \cdot [n(\widehat{Y}^d_{H,t})^2 + (1-n)(\widehat{Y}^d_{F,t})^2] - \frac{1}{2}\eta \cdot [n\widehat{Y}^2_{H,t} + (1-n)\widehat{Y}^2_{F,t}] \\ &\quad + \eta \cdot [n\widehat{Y}_{H,t}\overline{Y}^H_t + (1-n)\widehat{Y}_{F,t}\overline{Y}^F_t] + \\ &\quad - \frac{1}{2}(\sigma^{-1}+\eta) \cdot [n\mathrm{var}_h \widehat{y}_t(h) + (1-n)\mathrm{var}_f \widehat{y}_t(f)]\} \\ &\quad + \text{t.i.p.} + o(\|\xi\|^3), \end{aligned} \qquad \text{(E.20)}$$

and after substituting the expressions for $\widehat{Y}_{H,t}$, $\widehat{Y}_{F,t}$, $\widehat{Y}^d_{H,t}$, $\widehat{Y}^d_{F,t}$ we get

$$\begin{aligned}
w_t &= U_C\overline{C}\{\widehat{C}_t \cdot [n\Phi^H + (1-n)\Phi^F] + \frac{1}{2}(1-\rho)\widehat{C}_t^2 \\
&+\eta[\widehat{C}_t\overline{Y}_t^W + n(1-n)\widehat{T}_t\overline{Y}_t^R] - \frac{1}{2}[\widehat{C}_t^2 + n(1-n)\widehat{T}_t^2] \\
&-\frac{1}{2}\eta \cdot [\widehat{C}_t^2 + n(1-n)\widehat{T}_t^2 + 2\widehat{C}_t g_t^W - 2n(1-n)\widehat{T}_t g_t^R] \\
&-\frac{1}{2}(\sigma^{-1}+\eta) \cdot [n\text{var}_h \widehat{y}_t(h) + (1-n)\text{var}_f \widehat{y}_t(f)]\} \\
&+\text{t.i.p.} + o(\|\xi\|^3),
\end{aligned} \tag{E.21}$$

which can be written as

$$\begin{aligned}
w_t &= -U_C\overline{C}\{-\widehat{C}_t \cdot [n\Phi^H + (1-n)\Phi^F] \\
&+\frac{1}{2}(\rho+\eta)[\widehat{C}_t - \widetilde{C}_t]^2 + \frac{1}{2}(1+\eta)n(1-n)[\widehat{T}_t - \widetilde{T}_t]^2 \\
&+\frac{1}{2}(\sigma^{-1}+\eta) \cdot [n\text{var}_h \widehat{y}_t(h) + (1-n)\text{var}_f \widehat{y}_t(f)]\} \\
&+\text{t.i.p.} + o(\|\xi\|^3).
\end{aligned} \tag{E.22}$$

Where the natural rate of world consumption and of the term of trade, which will arise when prices are flexible, are defined as

$$\begin{aligned}
\widetilde{C}_t^W &\equiv \frac{\eta}{\rho+\eta}(\overline{Y}_t^W - g_t^W), \\
\widetilde{T}_t &\equiv \frac{\eta}{1+\eta}(g_t^R - \overline{Y}_t^R).
\end{aligned}$$

By using equations (E.13) and after having defined $c_t^W \equiv \widehat{C}_t^W - \widetilde{C}_t$ we obtain

$$\begin{aligned}
w_t &= -U_C\overline{C}\{\frac{1}{2}(\rho+\eta)[c_t^W - \overline{c}^W]^2 + \frac{1}{2}(1+\eta)n(1-n)[\widehat{T}_t - \widetilde{T}_t]^2 \\
&+\frac{1}{2}(\sigma^{-1}+\eta) \cdot [n\text{var}_h \widehat{y}_t(h) + (1-n)\text{var}_f \widehat{y}_t(f)]\} \\
&+\text{t.i.p.} + o(\|\xi\|^3),
\end{aligned} \tag{E.23}$$

where $\overline{c}^W \equiv -\ln\overline{C}/C^*$.

Here we derive $\text{var}_h \widehat{y}_t(h)$ and $\text{var}_f \widehat{y}_t(f)$. We have that

$$\text{var}_h\{\log y_t(h)\} = \sigma^2 \text{var}_h\{\log p_t(h)\}.$$

Defining $\overline{p}_t \equiv \text{E}_h \log p_t(h)$, we have

$$\begin{aligned}
\text{var}_h\{\log p_t(h)\} &= \text{var}_h\{\log p_t(h) - \overline{p}_{t-1}\} = \text{E}_h\{[\log p_t(h) - \overline{p}_{t-1}]^2\} - (\Delta\overline{p}_t)^2 \\
&= \alpha^H \text{E}_h\{[\log p_{t-1}(h) - \overline{p}_{t-1}]^2\} + (1-\alpha^H)[\log\widetilde{p}_t(h) - \overline{p}_{t-1}]^2 + \\
&\qquad -(\Delta\overline{p}_t)^2 \\
&= \alpha^H \text{var}_h\{\log p_{t-1}(h)\} + (1-\alpha^H)[\log\widetilde{p}_t(h) - \overline{p}_{t-1}]^2 - (\Delta\overline{p}_t)^2.
\end{aligned}$$

We have also that

$$\overline{p}_t - \overline{p}_{t-1} = (1-\alpha^H)[\log\widetilde{p}_t(h) - \overline{p}_{t-1}], \tag{E.24}$$

from which we obtain that

$$\mathrm{var}_h\{\log p_t(h)\} = \alpha^H \mathrm{var}_h\{\log p_{t-1}(h)\} + \frac{\alpha^H}{1-\alpha^H}(\Delta\overline{p}_t)^2.$$

But

$$\overline{p}_t = \log P_{H,t} + o(\|\xi\|^2),$$

which implies

$$\mathrm{var}_h\{\log p_t(h)\} = \alpha^H \mathrm{var}_h\{\log p_{t-1}(h)\} + \frac{\alpha^H}{1-\alpha^H}(\pi_t^H)^2 + o(\|\xi\|^3),$$

after integration of the above equation we obtain

$$\mathrm{var}_h\{\log p_t(h)\} = (\alpha^H)^{t+1}\mathrm{var}_h\{\log p_{-1}(h)\} + \sum_{s=0}^{t}(\alpha^H)^{t-s}\frac{\alpha^H}{1-\alpha^H}(\pi_t^H)^2 + o(\|\xi\|^3)$$

where we note that the first term in the right hand side is independent of the policy chosen after period $t \geq 0$. After taking the discounted value, with the discount factor β, we obtain

$$\sum_{t=0}^{\infty}\beta^t \mathrm{var}_h\{\log p_t(h)\} = \frac{\alpha^H}{(1-\alpha^H)(1-\alpha^H\beta)}\sum_{t=0}^{\infty}\beta^t(\pi_t^H)^2 + \text{t.i.p.} + o(\|\xi\|^3)$$

The same derivations apply also for the Foreign country. We define

$$\begin{aligned} d^H &\equiv \frac{\alpha^H}{(1-\alpha^H)(1-\alpha^H\beta)}, \\ d^F &\equiv \frac{\alpha^F}{(1-\alpha^F)(1-\alpha^F\beta)}. \end{aligned}$$

We can simplify (E.3) to

$$W_t = -\Omega\sum_{j=0}^{\infty}\beta^j L_{t+j} \tag{E.25}$$

where

$$L_{t+j} = \Lambda[c_{t+j}^W - \overline{c}^W]^2 + n(1-n)\Gamma[\widehat{T}_{t+j} - \widetilde{T}_{t+j}]^2 + \gamma(\pi_{t+j}^H)^2 + (1-\gamma)(\pi_{t+j}^F)^2 + \text{t.i.p.} + o(\|\xi\|^3),$$

which corresponds to equation (26) in the text, where $c^W = y^W$ and $\overline{y}^W \equiv \overline{c}^W$.

Furthermore

$$\begin{aligned}
\Omega &\equiv \frac{1}{2}U_C\overline{C}(nd^H+(1-n)d^F)\sigma(1+\sigma\eta) \\
\Lambda &\equiv \frac{k_C^H k_C^F}{\sigma}\frac{1}{nk_C^F+(1-n)k_C^H}, \\
\Gamma &\equiv \frac{k_T^H k_T^F}{\sigma}\frac{1}{nk_T^F+(1-n)k_T^H}, \\
\gamma &\equiv \frac{nd^H}{nd^H+(1-n)d^F}.
\end{aligned}$$

We note that when the degrees of rigidity are the same, i.e $d^H = d^F$, γ coincides with n.

Appendix E

In this appendix we sketch the main characteristic of the K-region extension. The whole economy is populated by a continuum of agents on the interval $[0, 1]$. Each agent is both consumer and producer. Consumer of all the goods produced within the economy, producer of a single differentiated product. In each sector a measure n_i of goods is produced, with $i = 1, 2, ..., K$. We have that $\sum_{i=1}^{K} n_i = 1$. Preferences of the generic household j are given by

$$U_t^j = \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left[U(C_s^j) + L\left(\frac{M_s^j}{P_s}, \xi^i \right) - V(y_s^j, z_s^i) \right],$$

where everything has the same interpretation as in the model of the main text, except that C^j is defined as

$$C^j \equiv \frac{\prod_{i=1}^{K} (C_i^j)^{n_i}}{\prod_{i=1}^{K} n_i}$$

and C_i^j is an index of goods produced in region i. Specifically,

$$C_i^j \equiv \left[\left(\frac{1}{n^i} \right)^{\frac{1}{\sigma}} \int_{u \in i} c^j(u)^{\frac{\sigma-1}{\sigma}} du \right]^{\frac{\sigma}{\sigma-1}}.$$

for $i = 1, 2, ..., K$. In this case we can write total demand of good h produced in region k as

$$y_k^d(h) = \left(\frac{p(h)}{P_k} \right)^{-\sigma} [(P_k^R)^{-1} C^W + G_k]$$

where the union aggregate consumption C^W is defined as

$$C^W \equiv \int_0^1 C^j dj,$$

and the relative price of region k with respect to the overall price index has been defined as $P_k^R \equiv P_k/P$ for $k = 1, 2, ..., K$. The supply side of the model is the same except that we have to deal with K regions.

Here we note that Lemma 1 can be extended to this general context by observing that the first order conditions are the same as in the previous case as well as the aggregate budget constraint of each region.

In the log-linear approximation we use the following notation. Given a generic variable X, a world variable X^W is defined as the weighted average of the region's variables with weights n_i

$$X^W \equiv \sum_{i=1}^{K} n_i X_i,$$

while a relative variable X_i^R is defined as

$$X_i^R \equiv X_i - X^W$$

while $X^R_{i,j}$ as

$$X^R_{i,j} \equiv X_i - X_j.$$

The flexible-price solution is

$$\begin{aligned}
\widetilde{C}^W_t &= \frac{\eta}{\rho+\eta}(\overline{Y}^W_t - g^W_t), \\
\widetilde{Y}^W_t &= \frac{\eta}{\rho+\eta}\overline{Y}^W_t + \frac{\rho}{\rho+\eta}g^W_t, \\
\widetilde{P}^R_{i,t} &= \frac{\eta}{1+\eta}(g^R_{i,t} - \overline{Y}^R_{i,t}).
\end{aligned}$$

Here we discuss how the log-linear approximation of the equilibrium will behave under the hypothesis of sticky prices. We obtain the log-linear version of the Euler equation and of aggregate outputs as

$$E_t\widehat{C}^W_{t+1} = \widehat{C}^W_t + \rho^{-1}(\widehat{R}_t - E_t\pi^W_{t+1}),$$

$$\widehat{Y}_{i,t} = -\widehat{P}^R_{i,t} + \widehat{C}^W_t + g^i_t,$$

for each $i = 1, 2, ...K$. Our set of AS equations will be

$$\pi^i_t = -k^i_T(\widehat{P}^R_{i,t} - \widetilde{P}^R_{i,t}) + k^i_C(\widehat{C}^W_t - \widetilde{C}^W_t) + \beta E_t\pi^i_{t+1}, \qquad \text{for } i = 1, 2...K$$

Furthermore the definitions of relative price imply

$$P^R_{i,t} = \widehat{P}^R_{i,t-1} + \pi^i_t - \pi^W_t, \qquad \text{for } i = 1, 2...K$$

The welfare criterion of the Central Bank is again the discounted value of a weighted average of the average utility flows of all the households,

$$W = \mathrm{E}_0\sum_{j=0}^{\infty}\sum_{i=1}^{K}\beta^j n_i w^i_{t+j}.$$

In this case we obtain

$$\begin{aligned}
w^i_t = {} & U_C\overline{C}[\widehat{C}^i_t + \frac{1}{2}(1-\rho)(\widehat{C}^i_t)^2 - (1-\Phi)\cdot\widehat{Y}^d_{i,t} - \frac{1}{2}\cdot[\widehat{Y}^d_{i,t}]^2 - \frac{\eta}{2}\cdot[\widehat{Y}_{i,t}]^2 \\
& -\frac{1}{2}(\sigma^{-1}+\eta)\cdot\mathrm{var}_h\widehat{y}_{i,t}(h) + \eta\widehat{Y}^d_{i,t}\overline{Y}^i_t] + \text{t.i.p.} + o(\|\xi\|^3).
\end{aligned} \tag{F.1}$$

Taking a linear combination of (F.1) with weights n_i, we have

$$\begin{aligned}
w_t = {} & U_C\overline{C}\{\widehat{C}^W_t\cdot[\Phi] + \frac{1}{2}(1-\rho)(\widehat{C}^W_t)^2 \\
& -\frac{1}{2}\cdot[\sum_{i=1}^{K}n_i(\widehat{Y}^d_{i,t})^2] - \frac{1}{2}\eta\cdot[\sum_{i=1}^{K}n_i(\widehat{Y}_{i,t})^2] \\
& +\eta\cdot[\sum_{i=1}^{K}n_i\widehat{Y}_{i,t}\overline{Y}^i_t] - \frac{1}{2}(\sigma^{-1}+\eta)\cdot[\sum_{i=1}^{K}n_i\mathrm{var}_h\widehat{y}_{i,t}(h)]\} \\
& +\text{t.i.p.} + o(\|\xi\|^3),
\end{aligned}$$

Noting that

$$\begin{aligned}
\sum_{i=1}^{K} n_i(\widehat{Y}^d_{i,t})^2 &= (\widehat{C}^W_t)^2 + \sum_{i=1}^{K} n_i(\widehat{P}^R_{i,t})^2, \\
\sum_{i=1}^{K} n_i(\widehat{Y}_{i,t})^2 &= (\widehat{C}^W_t)^2 + \sum_{i=1}^{K} n_i(\widehat{P}^R_{i,t})^2 - 2\sum_{i=1}^{K} n_i\widehat{P}^R_{i,t}g_{i,t} + 2\widehat{C}^W_t g^W_t, \\
\sum_{i=1}^{K} n_i\widehat{Y}_{i,t}\overline{Y}^i_t &= \widehat{C}^W_t\overline{Y}^W_t - \sum_{i=1}^{K} n_i\widehat{P}^R_{i,t}\overline{Y}^i_t, \\
\sum_{i=1}^{K} n_i\widehat{P}^R_{i,t} &= 0,
\end{aligned}$$

we obtain

$$\begin{aligned}
w_t &= -U_C\overline{C}\{\frac{1}{2}(\rho+\eta)[c^W_t - \overline{c}^W]^2 + \frac{1}{2}(1+\eta)[\sum_{i=1}^{K} n_i(\widehat{P}^R_{i,t} - \widetilde{P}^R_{i,t})^2] + \\
&\quad + \frac{1}{2}(\sigma^{-1}+\eta)\cdot[\sum_{i=1}^{K} n_i \mathrm{var}_h \widehat{y}_{i,t}(h)]\} + \text{t.i.p.} + o(\|\xi\|^3),
\end{aligned} \tag{F.2}$$

where $\overline{c}^W \equiv -\ln\overline{C}/C^*$.

Moreover we have that

$$\sum_{t=0}^{\infty}\beta^t var_h\widehat{y}_{i,t}(h) = \sigma^2\frac{\alpha^i}{(1-\alpha^i)(1-\alpha^i\beta)}\sum_{t=0}^{\infty}\beta^t(\pi^i_t)^2 + t.i.p. + o(\|\xi\|^3).$$

Defining

$$d^i \equiv \frac{\alpha^i}{(1-\alpha^i)(1-\alpha^i\beta)},$$

we can simplify the welfare function to

$$W_t = -\Omega\sum_{j=0}^{\infty}\beta^j L_{t+j} \tag{F.3}$$

where

$$L_{t+j} = \Lambda[c^W_{t+j} - \overline{c}^W]^2 + \Gamma[\sum_{i=1}^{K} n_i(\widehat{P}^R_{i,t} - \widetilde{P}^R_{i,t})^2] + \sum_{i=1}^{K}\gamma_i(\pi^i_{t+j})^2 + \text{t.i.p.} + o(\|\xi\|^3),$$

and

$$
\begin{aligned}
\Omega &\equiv \frac{1}{2} U_C \overline{C} (\sum_{i=1}^{K} n_i d^i) \sigma (1 + \sigma\eta) \\
\Lambda &\equiv \frac{1}{\sigma} \left(\sum_{i=1}^{K} n_i (k_C^i)^{-1} \right)^{-1} \\
\Gamma &\equiv \frac{1}{\sigma} \left(\sum_{i=1}^{K} n_i (k_T^i)^{-1} \right)^{-1}, \\
\gamma_i &\equiv \frac{n_i d^i}{(\sum_{i=1}^{K} n_i d^i)}.
\end{aligned}
$$

We note that when the degrees of rigidity are the same, γ_i coincides with n_i.

Given this structure, some generalizations of the results of the main text follow directly. Efficiency can be obtained only if $K - 1$ regions have flexible prices. In this case monetary policy should target the inflation rate in the sticky price region. If all the regions have the same degree of nominal rigidity, then the optimal policy is to target to zero π_t^W. If we restrict the attention to the inflation targeting class of policies, regions with equal degree of nominal rigidity should have the same weight. For example, if only one sector has flexible prices, and the others have identical degree of nominal rigidity, then it is optimal to target a weighted average of the sticky-price inflations with equal weights.