Supplementary Appendix[1] to:
"Linear-Quadratic Approximation of Optimal Policy Problems"
by Pierpaolo Benigno and Michael Woodford

This appendix provides further details on the solution method that can be helpful to facilitate its implementation.

Solution to the first-order conditions

The first-order conditions, which are discussed in section 3 of the paper, are given by

$$\frac{1}{2}E_t\{[A(L)+A'(\beta L^{-1})]\tilde{y}_t+E_t[B(L)\xi_{t+1}]+E_t[C'(\beta L^{-1})\tilde{\lambda}_t]+\beta^{-1}D'(\beta L^{-1})\tilde{\varphi}_{t-1}=0, \tag{1}$$

$$E_t D(L)\tilde{y}_{t+1} = D_2\xi_t, \tag{2}$$

$$C(L)\tilde{y}_t = C_2\xi_t, \tag{3}$$

which have to be solved for the joint evolution of the processes $\{\tilde{y}_t, \tilde{\lambda}_t, \tilde{\varphi}_t\}$ given the exogenous disturbance processes $\{\xi_t\}$ and the initial conditions $\tilde{y}_{t_0-1}$ and $\tilde{\varphi}_{t_0-1}$. Let assume that the exogenous disturbance processes follow

$$\xi_t = \Theta\xi_{t-1} + \Lambda\epsilon_t, \tag{4}$$

where ϵ_t has zero mean and variance-covariance matrix given by the identity matrix. These first-order conditions can be written as

$$\begin{aligned}&\frac{\beta}{2}A_1E_t\tilde{y}_{t+1} + (B_0\Theta + B_1)\xi_t + \beta C_1'E_t\tilde{\lambda}_{t+1} + D_1'\tilde{\varphi}_t\\ = \quad & -\frac{1}{2}(A_0 + A_0')\tilde{y}_t - \frac{1}{2}A_1\tilde{y}_{t-1} - C_0'\tilde{\lambda}_t - \beta^{-1}D_0'\tilde{\varphi}_{t-1} - B_2\xi_{t-1},\end{aligned} \tag{5}$$

$$D_0E_t\tilde{y}_{t+1} = -D_1\tilde{y}_t + D_2\xi_t, \tag{6}$$

$$C_0\tilde{y}_t = -C_1\tilde{y}_{t-1} + C_2\xi_t, \tag{7}$$

where we have defined

$$A_0 \equiv Q$$

$$A_1 \equiv 2R$$

$$B_0 \equiv \beta\bar{\lambda}_k D^2_{\tilde{y}\xi}F^k$$

$$B_1 \equiv (\bar{\lambda}_k D^2_{y\xi}F^k + \bar{\varphi}_i D^2_{y\xi}g^i + D^2_{y\xi}\pi)$$

$$B_2 \equiv \beta^{-1}\bar{\varphi}_i D^2_{\tilde{y}\xi}g^i$$

$$C_0 \equiv D_yF$$

$$C_1 \equiv D_{\tilde{y}}F$$

[1]We are grateful to Vasco Curdia for helpful comments.

$$C_2 \equiv -D_\xi F$$
$$D_0 \equiv D_{\hat{y}} g$$
$$D_1 \equiv D_y g$$
$$D_2 \equiv -D_\xi g$$

for matrices and notation detailed in section 2.2 of the paper. First-order conditions (5) to (7) together with the processes (4) can be written in a system of the form

$$G_0 E_t u_{t+1} = G_1 u_t + G_2 \xi_t + G_3 \xi_{t-1} \tag{8}$$

where the vector u_t is defined as

$$u_t \equiv \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix}$$

and z_t is defined as

$$z_t \equiv \begin{bmatrix} \tilde{\lambda}_t \\ \tilde{y}_t \\ \tilde{\varphi}_t \end{bmatrix}$$

The matrices are given by

$$G_0 \equiv \begin{bmatrix} \beta C_1' & \frac{\beta}{2} A_1 & 0 & C_0' & \frac{1}{2}(A_0 + A_0') & D_1' \\ 0 & D_0 & 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 & C_0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$G_1 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} A_1 & -\beta^{-1} D_0' \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -C_1 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix},$$

$$G_2 \equiv \begin{bmatrix} -(B_0 \Theta + B_1) \\ D_2 \\ C_2 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$G_2 \equiv \begin{bmatrix} -B_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The characteristic polynomial of the system (8) is given by

$$\det[\mu G_0 - G_1] = 0$$

and the above system is determined if there are n (where n is the dimension of the vector z) roots within the unit circle. Let us call G_e the matrices of left eigenvectors associated with the unstable eigenvalues that satisfies

$$\Omega_e G_e G_1 = G_e G_0,$$

where Ω_e is a triangular stable matrix of dimension n. We can then pre-multiply (8) with the product of matrices $\Omega_e G_e$ and obtain

$$\Omega_e E_t \bar{u}_{t+1} = \bar{u}_t + \Omega_e G_e G_2 \xi_t + \Omega_e G_e G_3 \xi_{t-1},$$

where we have defined $\bar{u}_t \equiv G_s u_t$ where $G_s \equiv G_e G_0$. This equation has a stable solution of the form

$$\bar{u}_t = -\Omega_e E_t \left\{ \sum_{T=t}^{+\infty} \Omega_e{}^{(T-t)} (G_e G_2 + \Omega_e G_e G_3) \xi_T \right\} - \Omega_e G_e G_3 \xi_{t-1}. \qquad (9)$$

We can partition G_s as $G_s = [G_a \; G_b]$ where G_a and G_b are square matrices of dimension n. Using this partition, the definition of $\bar{u}_t$ and (4) it follows that

$$z_t = \bar{\Psi}_0 z_{t-1} + \bar{\Psi}_1 \xi_t + \bar{\Psi}_2 \xi_{t-1} \qquad (10)$$

where

$$\bar{\Psi}_0 = -G_a^{-1} G_b$$
$$\bar{\Psi}_1 = -G_a^{-1} \Omega_e \bar{U}$$
$$\bar{\Psi}_2 = -G_a^{-1} \Omega_e G_e G_3$$

In particular $\bar{U}$ solves

$$\bar{U} = (G_e G_2 + \Omega_e G_e G_3) + \Omega_e \bar{U} \Theta$$

which implies in a closed-form solution that

$$vec(\bar{U}) = [I - (\Theta' \otimes \Omega_e)]^{-1} vec[G_e G_2 + \Omega_e G_e G_3].$$

Since $\tilde{\lambda}_{t-1}$ does not enter the first-order conditions then $\bar{\Psi}_0$ is triangular in way that it is possible to extract from (10) the law of motion

$$\begin{bmatrix} \tilde{y}_t \\ \tilde{\varphi}_t \end{bmatrix} = \tilde{T} \begin{bmatrix} \tilde{y}_{t-1} \\ \tilde{\varphi}_{t-1} \end{bmatrix} + \Psi_1 \xi_t + \Psi_2 \xi_{t-1}$$

for selected matrices $\tilde{T}$, Ψ_1 and Ψ_2.This allows to write the evolution of the state vector

$$\mathbf{y}_t \equiv \begin{bmatrix} \tilde{y}_t \\ \tilde{\varphi}_t \\ \xi_t \end{bmatrix}$$

as in the appendix of the paper. Indeed, we have

$$\mathbf{y}_t = \Sigma \mathbf{y}_{t-1} + \Xi \epsilon_t$$

where

$$\Sigma \equiv \begin{bmatrix} \tilde{T} & \Psi_0 \Theta + \Psi_1 \\ 0 & \Theta \end{bmatrix}$$

$$\Xi \equiv \begin{bmatrix} \Psi_1 \Lambda \\ \Lambda \end{bmatrix}$$

Second-order conditions

The second-order conditions are based on matrices given in Lemma 2 of the paper. In particular the matrix T can be obtained from $\tilde{T}$ computed above as $T = \beta^{\frac{1}{2}} \tilde{T}$. The matrix J is implicitly defined by

$$J = T'[S'(A_0 + A_0')S + \beta^{1/2} T' S' A_1 S + \beta^{1/2} S' A_1' S T]T + T' J T$$

and can be obtained by solving

$$vec(J) = [I - (T' \otimes T')]^{-1} vec(T'[S'(A_0 + A_0')S + \beta^{1/2} T' S' A_1 S + \beta^{1/2} S' A_1' S T]T).$$

Evaluation of alternative policy rules

To evaluate alternative policy rules, it is useful to add constraints that are not binding in the optimal policy problem, since the evolution of additional endogenous variables might be needed for the purpose of evaluating alternative policy rules.

Under optimal policy, we have shown that the state vector evolves according to

$$\mathbf{y}_t = \Sigma \mathbf{y}_{t-1} + \Xi \epsilon_t. \tag{11}$$

However it might be possible that additional lags of $\tilde{y}_t$ and ξ_t are needed because they are part of the state vector implied by the alternative policy rules. So the vector $\mathbf{y}_t$ should be appropriately extended and matrices Σ and Ξ modified appropriately. Once the definitions of $\mathbf{y}_t$ agree between the optimal policy and the alternative regime we can write the evolution of the state vector in the latter case as

$$\mathbf{y}_t = \Sigma^r \mathbf{y}_{t-1} + \Xi^r \epsilon_t, \tag{12}$$

for appropriate matrices Σ^r and Ξ^r.

First, we are interested in the decomposition of the state vector into a cyclical and trend component. Let assume generically that the state vector of dimension k is non-stationary, but difference stationary. Assume that there are p cointegrating vectors, with $p \leq k$. It follows that we can write

$$\Delta \mathbf{y}_t = \Upsilon \mathbf{y}_{t-1} + \Xi \epsilon_t,$$

where $\Upsilon \equiv \Sigma - I$ which can be decomposed as $\Upsilon = \Upsilon_\alpha \Upsilon'_\beta$ where Υ_α and Υ_β are both $k \times p$ matrices and in particular $\Upsilon'_\beta \mathbf{y}_t = \mathbf{c}_t$ where $\mathbf{c}_t$ is the cointegrating vector of dimension p. Note that we can write

$$\Delta \mathbf{y}_t = \Upsilon_\alpha \mathbf{c}_{t-1} + \Xi \epsilon_t \tag{13}$$

and

$$\mathbf{c}_t = (I + \Upsilon'_\beta \Upsilon_\alpha)\mathbf{c}_{t-1} + \Upsilon'_\beta \Xi \epsilon. \tag{14}$$

Since there is no drift, we can define the trend component as

$$\mathbf{y}_t{}^{tr} = \lim_{T \longrightarrow \infty} E_t \mathbf{y}_T = \mathbf{y}_t + E_t \sum_{j=1}^{\infty} \Delta \mathbf{y}_{t+j}$$

It follows that

$$\mathbf{y}_t{}^{cyc} = -E_t \sum_{j=1}^{\infty} \Delta \mathbf{y}_{t+j}.$$

Using (13) and (14) we can write

$$\mathbf{y}_t{}^{cyc} = \Upsilon_\alpha (\Upsilon'_\beta \Upsilon_\alpha)^{-1} \mathbf{c}_t,$$

while the trend component can be written as

$$\mathbf{y}_t{}^{tr} = (I - \Upsilon_\alpha (\Upsilon'_\beta \Upsilon_\alpha)^{-1} \Upsilon'_\beta) \mathbf{y}_t$$

where P, defined in the appendix of the paper, is given by

$$P \equiv I - \Upsilon_\alpha (\Upsilon'_\beta \Upsilon_\alpha)^{-1} \Upsilon'_\beta.$$

Let us define $\Upsilon_c \equiv \Upsilon_\alpha (\Upsilon'_\beta \Upsilon_\alpha)^{-1}$ we can then write

$$\mathbf{y}_t{}^{cyc} = \Upsilon_c \mathbf{c}_t.$$

The variance-covariance matrix of the cyclical component then solves

$$\mathbf{V} = \Upsilon_c V_c \Upsilon'_c$$

where

$$V_c = \Lambda_c V_c \Lambda'_c + \Upsilon'_\beta \Xi \Xi' \Upsilon_\beta$$

and we have defined $\Lambda_c = (I + \Upsilon'_\beta \Upsilon_\alpha)$. Note that this has a closed-form solution of the form

$$vec(V_c) = [I - (\Lambda_c \otimes \Lambda_c)]^{-1} \{vec \Upsilon'_\beta \Xi \Xi' \Upsilon_\beta\}.$$

Note also that

$$\mathbf{y}_t{}^{cyc} = \Upsilon_\alpha (\Upsilon'_\beta \Upsilon_\alpha)^{-1} \Upsilon'_\beta \mathbf{y}_t = (I - P)\mathbf{y}_t$$

from which it follows that

$$\mathbf{y}_t{}^{cyc} = \Sigma \mathbf{y}_{t-1}{}^{cyc} + (I - P)\Xi \epsilon_t,$$

where Σ is the same matrix as in (11). The above decomposition applies also to the vector $\mathbf{y}_t$ following the law of motion (12). However matrices, cointegrating vectors and number of cointegrating vectors are not necessarily the same.

Note that both under optimal policy and under alternative regimes we have

$$\tilde{y}_t = S_y \mathbf{y}_t$$

$$\tilde{\varphi}_t = S_\varphi \mathbf{y}_t$$

$$\xi_t = S_\xi \mathbf{y}_t$$

where S_y, S_φ and S_ξ are selection matrices that select the appropriate parts of the state vector.

The objective of interest for the evaluation of alternative policy rules is equation (4.2) in the paper

$$\begin{aligned} W_{t_0} &= \frac{1}{2} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\tilde{y}_t' A_0 \cdot \tilde{y}_t + \tilde{y}_t' A_1 \cdot \tilde{y}_{t-1} + 2\tilde{y}_t' [B_0 \Theta + B_1] \cdot \xi_t + 2\tilde{y}_t' B_2 \cdot \xi_{t-1} \right] + \\ &\quad + \beta^{-1} \tilde{\varphi}_{t_0-1}' D_0 \cdot \tilde{y}_{t_0} \end{aligned}$$

that can be written as

$$\begin{aligned} W_{t_0} &= \frac{1}{2} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} tr\{[S_y' A_0 S_y + 2S_y'(B_0 \Theta + B_1) S_\xi] \cdot \mathbf{y}_t \mathbf{y}_t'\} + tr\{[S_y' A_1 S_y + 2S_y' B_2 S_\xi] \cdot \mathbf{y}_{t-1} \mathbf{y}_t'\} + \\ &\quad + \beta^{-1} tr\{S_\varphi' D_0 S_y \cdot \mathbf{y}_{t_0} \mathbf{y}_{t_0-1}'\}. \end{aligned}$$

As discussed in the main text we are interested in evaluating $E_\mu E_{t_0-1} W_{t_0}$ where E_μ is the expectation computed using the invariant distribution of ${\mathbf{y}_t}^{cyc}$ under the optimal policy. Note that we can write

$$\mathbf{y}_t = {\bar{\mathbf{y}}_t}^{tr} + {\bar{\mathbf{y}}_t}^{cyc} + \mathbf{y}_t^\dagger$$

where ${\bar{\mathbf{y}}_t}^{tr} \equiv E_{t_0-1} {\mathbf{y}_t}^{tr}$ and ${\bar{\mathbf{y}}_t}^{cyc} \equiv E_{t_0-1} {\mathbf{y}_t}^{cyc}$ and $\mathbf{y}_t^\dagger = \mathbf{y}_t - E_{t_0-1} \mathbf{y}_t$. Moreover

$$\begin{aligned} {\bar{\mathbf{y}}_t}^{tr} &= \Sigma^r {\bar{\mathbf{y}}_{t-1}}^{tr}, && (15) \\ {\bar{\mathbf{y}}_t}^{cyc} &= \Sigma^r {\bar{\mathbf{y}}_{t-1}}^{cyc}, && (16) \\ \mathbf{y}_t^\dagger &= \Sigma^r \mathbf{y}_{t-1}^\dagger + \Xi^r \epsilon_t. && (17) \end{aligned}$$

We define the object

$$\mathcal{P}(\mathbf{y}_{t_0-1}) = (1-\beta) E_{t_0-1} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \mathbf{y}_{t-1} \mathbf{y}_{t-1}'.$$

Given (17) we note that

$$E_{t_0-1} \mathcal{P}(\mathbf{y}_{t_0}^\dagger) = \Xi^r \Xi^{r\prime} + \beta \Sigma^r \{E_{t_0-1} \mathcal{P}(\mathbf{y}_{t_0}^\dagger)\} \Sigma^{r\prime}$$

and then

$$vec[E_{t_0-1}\mathcal{P}(\mathbf{y}_{t_0}^{\dagger})] = [I - (\beta\Sigma^r \otimes \Sigma^r)]^{-1} vec[\Xi^r \Xi^{r\prime}].$$

Note that

$$\mathcal{P}(\bar{\mathbf{y}}_{t_0-1}{}^{cyc}) = (1-\beta)E_{t_0-1}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\bar{\mathbf{y}}_{t-1}{}^{cyc}\bar{\mathbf{y}}'_{t-1}{}^{cyc}.$$

Given (16) we obtain

$$E_\mu\mathcal{P}(\bar{\mathbf{y}}_{t_0-1}{}^{cyc}) = (1-\beta)E_\mu\{\mathbf{y}_{t_0-1}{}^{cyc}\mathbf{y}'_{t_0-1}{}^{cyc}\} + \beta\Sigma^r[E_\mu\mathcal{P}(\bar{\mathbf{y}}_{t_0-1}{}^{cyc})]\Sigma^{r\prime}$$

and in a closed-form solution

$$vec[E_\mu\mathcal{P}(\bar{\mathbf{y}}_{t_0-1}{}^{cyc})] = [I - (\beta\Sigma^r \otimes \Sigma^r)]^{-1} vec[(1-\beta)\mathbf{V}],$$

since E_μ is taken across the stationary distribution of the cyclical component implied by the optimal policy.

Note that the object

$$\mathcal{P}(\mathbf{y}_{t_0}) = (1-\beta)E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbf{y}_t\mathbf{y}'_t$$

is such that

$$E_\mu E_{t_0-1}\mathcal{P}(\mathbf{y}_{t_0}) = \Sigma^r\mathcal{P}(\bar{\mathbf{y}}_{t_0-1}{}^{tr})\Sigma^{r\prime} + E_{t_0-1}\mathcal{P}(\mathbf{y}_{t_0}^{\dagger}) + \Sigma^r[E_\mu\mathcal{P}(\bar{\mathbf{y}}_{t_0-1}{}^{cyc})]\Sigma^{r\prime}$$

where $\mathcal{P}(\bar{\mathbf{y}}_{t_0-1}{}^{tr})$ is independent of policy since both the optimal policy and the alternative rules have the same decomposition between trend and cyclical component at time t_0-1. (as it is discussed in section 4 of the paper) Moreover defining

$$\mathcal{F}(\mathbf{y}_{t_0}) = (1-\beta)E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\mathbf{y}_{t-1}\mathbf{y}'_t,$$

we note that

$$E_\mu E_{t_0-1}\mathcal{F}(\mathbf{y}_{t_0}) = \mathcal{P}(\bar{\mathbf{y}}_{t_0-1}{}^{tr})\Sigma^{r\prime} + \beta[E_{t_0-1}\mathcal{P}(\mathbf{y}_{t_0}^{\dagger})]\Sigma^{r\prime} + [E_\mu\mathcal{P}(\bar{\mathbf{y}}_{t_0-1}{}^{cyc})]\Sigma^{r\prime}.$$

Finally

$$\begin{aligned} E_\mu E_{t_0-1}\{\mathbf{y}_{t_0}\mathbf{y}'_{t_0-1}\} &= \Sigma^r\{\bar{\mathbf{y}}_{t_0-1}{}^{tr}\bar{\mathbf{y}}'_{t_0-1}{}^{tr}\} + \Sigma^r E_\mu\{\mathbf{y}_{t_0-1}{}^{cyc}\mathbf{y}'_{t_0-1}{}^{cyc}\} \\ &= \Sigma^r\{\bar{\mathbf{y}}_{t_0-1}{}^{tr}\bar{\mathbf{y}}'_{t_0-1}{}^{tr}\} + \Sigma^r\mathbf{V}. \end{aligned}$$

We can then evaluate the welfare obtaining

$$\begin{aligned} E_\mu\bar{W} &= \frac{1}{2(1-\beta)}tr\{[S'_y A_0 S_y + 2S'_y(B_0\Theta + B_1)S_\xi]\cdot[E_{t_0-1}\mathcal{P}(\mathbf{y}_{t_0}^{\dagger}) + \Sigma^r(E_\mu\mathcal{P}(\bar{\mathbf{y}}_{t_0-1}{}^{cyc}))\Sigma^{r\prime}]\} \\ &\quad + \frac{1}{2(1-\beta)}tr\{[S'_y A_1 S_y + 2S'_y B_2 S_\xi]\cdot[\beta(E_{t_0-1}\mathcal{P}(\mathbf{y}_{t_0}^{\dagger}))\Sigma^{r\prime} + (E_\mu\mathcal{P}(\bar{\mathbf{y}}_{t_0-1}{}^{cyc}))\Sigma^{r\prime}]\} + \\ &\quad + \beta^{-1}tr\{S'_\varphi D_0 S_y\cdot\Sigma^r\mathbf{V}\} + \bar{W}(\bar{\mathbf{y}}_{t_0-1}{}^{tr}), \end{aligned}$$

where $\bar{W}(\bar{\mathbf{y}}_{t_0-1}{}^{tr})$ is independent of policy as discussed in section 4 of the paper.